A bijection between the d-dimensional simplices with distances in $\{1,2\}$ and the partitions of $d+1$

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Integral point sets are sets of n points in the Euclidean space $\mathbb{E}^d$ with integral distances between vertices, see [3] for a survey. We examined such point sets for $n = d+1$ and received the following table of numbers of nonisomorphic integral simplices by computer calculations. Here we call the largest occurring distance the diameter of the point set. Due to the triangle inequality the (d+1)-element vertex set

diameter	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
1	1	1	1	1	1	1	1
2	4	6	10	14	21	29	41
3	16	56	197	656	2127	6548	19130
4	45	336	3133	31771	329859	3336597	32815796

Table 1. Number of nonisomorphic integral simplices by diameter and dimension.

is partitioned into subsets of vertices having pairwise distance 1, whereas vertices of different subsets are at distance 2. To prove the proposed bijection, we have to provide a simplex for a given partition $(n_1, \ldots, n_r)$ of $d+1$. We would like to mention that the bijection holds more generally for simplices with distances in $\{1, \lambda\}$ for $\lambda \geq 2$. At first we give the following explicit construction.

Construction. Place regular (n_i-1)-simplices with edge length 1 with their barycenters at the origin into mutually orthogonal spaces. Then shift the i^{th} simplex into a new coordinate direction by the amount of $\sqrt{\frac{\lambda^2}{2} - \frac{n_i-1}{2n_i}}$.

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For another proof we need the following criterion.

Theorem (Menger [6]). If M is a set of $d+1$ points with distance matrix $D = (d_{i,j})$ and $A = (d_{i,j}^2)$, then M is realizable in the Euclidean d-dimensional space, iff $(-1)^{d+1}\det(\overline{A}) \geq 0$ and each subset of M is realizable in the $(d-1)$-dimensional space, where $\overline{A} := \begin{pmatrix} 0 & (1,\ldots,1)^T \\ (1,\ldots,1) & A \end{pmatrix}$. To apply this theorem we provide, for distance matrices derived from a partition of $d+1$ and with the nonzero values being in $\{1,\lambda\}$, the following lemma.

Lemma. $\quad (-1)^{d+1}(\lambda^2 \det(\overline{A}) + \det(A)) > 0 \quad$ and $\quad (-1)^{d+1}\det(\overline{A}) > 0.$

We leave the proof to the reader, because it can be easily but lengthly done by induction on d. It should be remarked that a formula for the relevant determinants was also stated in [4], but with no details of the computation.

As a last remark we would like to mention that using [5] one can generalize the stated bijection. For given d only $\lambda \geq \sigma(d-1, d+1)$ is needed, where

$$\sigma(d, d+2) = \sqrt{\frac{9d - 10 + \sqrt{33d^2 - 52d + 20}}{4d-4}} \geq \frac{1}{2}\sqrt{9 + \sqrt{33}} \approx 1.91993.$$

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