On the Democratic Weights of Nations

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Abstract

Which voting weights ought to be allocated to single delegates of differently sized groups from a democratic fairness perspective? We operationalize the 'one person, one vote' principle by demanding every individual's influence on collective decisions to be equal a priori. The analysis differs from previous ones by considering intervals of alternatives. New reasons lead to an old conclusion: weights should be proportional to the square root of constituency sizes if voter preferences are independent and identically distributed. This finding is fragile, however, in that preference polarization along constituency lines quickly calls for plain proportionality.

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I. Introduction

In decision-making bodies with a divisional or regional structure, the sizes or voting weights of different delegations commonly vary in the numbers of represented constituents. How they do so varies, too. In the US Electoral College, for instance, each state has two votes in addition to a number which is proportional to population size. California and Wyoming with around 11.9% and 0.2% of the US population thus end up holding around 10.2% and 0.6% of votes on the US President. In contrast, the most and least populous member states of the EU – Germany and Malta – currently have about 8.2% and 0.9% of votes in the Council of the European Union but comprise 15.9% and 0.1% of the EU population; the respective mapping from population size to voting weight is, very roughly, a square root function.[1] Delegates in other decision-making bodies, such as the Senate of Canada, the Assembly of the African Union, the Governing Council of the European Central Bank and many a university senate or council of a multi-branch NGO, have voting weights that are yet more concave in the number of represented individuals, or even flat.

This paper analyzes democratic fairness of different voting weight arrangements. Individuals choose delegates in disjoint constituencies (bottom tier) and these representatives take collective decisions in an assembly (top tier). We investigate a practical question: which mapping – possibly linear, possibly strictly concave or constant – should determine the top-tier voting weights of delegates from differently sized constituencies? Barberà and Jackson (2006) and Koriyama et al. (2013), among others, have studied this question for binary policy spaces, with the objective to maximize a utilitarian welfare function. We focus on the basic democratic principle of *'one person, one vote'*. Our corresponding conception of equitable institutional design is that all bottom-tier voters should wield

[1]A least squares power-law regression of voting weights w_i in the EU's Treaty of Nice on population sizes n_i results in $w_i = c \cdot n_i^{0.48}$ with $R^2 \approx 0.95$. The corresponding voting rule can be invoked until 2017 and will then be replaced by the more proportional system agreed in the Treaty of Lisbon.

equal influence on collective decisions – at least under stylized ideal conditions behind a constitutional veil of ignorance.

The major difference to the existing literature is that agents face an interval of policy alternatives, rather than binary ones. This opens up the analysis to a collection of economic issues (such as tax rates, monetary policy, spending on climate change mitigation) that otherwise would not be covered. We assume that voter preferences are single-peaked. Bargaining, political competition and other types of interaction at the constituency level can then be captured in reduced form by considering the respective *median voter*. Specifically, the realized median preference of each constituency is presumed to act as its representative. The representatives then use a weighted voting rule with a 50% majority threshold and adopt the *Condorcet winner* among their ideal points, i.e., the policy which beats all alternatives in a pair-wise vote. This coincides, formally, with the weighted median in the assembly.

The distributions of voter preferences and the delegates' voting weights jointly determine how often a given constituency is pivotal – that is, how often its most preferred policy is the assembly's weighted median. Our design objective calls for weights to be such that each constituency's probability of being pivotal is *proportional to its population size*, making the a priori assumption that voter preferences are all identically distributed as well as mutually independent across constituencies.

Proportionality of pivot probabilities at the top tier is linked to equal individual influence in two ways. The first is to view an individual citizen's influence as his or her chances to induce a collective outcome in accordance with the personal preference's ideal point: that is, to be a median voter of the decisive constituency. The probability of being the local median is inversely proportional to the constituency's population size. If this is balanced by proportional pivot probabilities for the representatives, the democratic playing field is level.

Alternatively, one can identify a constituent's influence with the anticipated effect of taking part in the decision making process. Even if the constituency's preferences coincide with those of only one individual, *every* constituency member affects who this is. If, say, some left-wing voter dropped out – so an ideal point to the left of the median is deleted from the constituency's preference distribution – then the median's location would shift to the right. The expected size of such shift and hence the influence associated with democratic participation are inversely proportional to constituency size. Again, proportionality at the top tier is required in order to avoid bias.

The relation of *heterogeneity within each constituency* and *heterogeneity across constituencies* turns out to be the critical parameter for a fair weight allocation, because it determines how the probability distributions of the constituency medians vary with population sizes. A greater number of preference draws generally reduces the variance of the resulting median. The representative of a large constituency will therefore frequently hold a central political position, at least if all preferences are independent. This raises the odds of being the assembly's median compared to delegates from small constituencies, who are more often outliers. In view of this size-related advantage of large constituencies, strictly concave or 'degressively proportional' weights can be enough to induce proportionality of pivot probabilities in the assembly. However, positive correlation within the constituencies – corresponding to local fixed effects and reflecting heterogeneity across constituencies – slows the reaction of variance to population size. If, consequently, representatives of large constituencies have little or no locational advantage in the assembly, they need more progressive margins for their voting weights.

Square root and linear weighting rules emerge from these considerations in important benchmark cases. The former is advisable if all voter preferences are mutually independent (Corollary 1), while the considered democratic ideal calls for linear allocations when there is sufficient heterogeneity across constituencies, i.e., if voters are polarized along constituency lines (Corollary 2). These conclusions follow from a new limit result on committee decisions on intervals, which may also find other applications (Theorem 1).

Of course, an individual's probability of being pivotal and also the expected policy effects of participation will be very small for most real-life population figures.[2] In relative terms, however, these probabilities can differ widely across constituencies when weights are chosen arbitrarily. They should not, first, as a matter of democratic principle. Second, though we make no cardinal assumptions in the analysis, large issues can be at stake. The associated variation of expected utilities may hence be significant. One explanation of why people vote at all is that they care about the general good, i.e., they have social preferences. If election of candidate X rather than Y, or a policy shift to the left or right, would raise quality of life by the equivalent of $100 for ten million fellow citizens in the eyes of a given, socially-minded individual, then tiny prospects of affecting the outcome become billion-dollar lotteries; their allocation matters (see Edlin et al. 2007). Finally, variation of pivot probabilities makes it profitable for politicians to concentrate their attention and resources, i.e., target policies to states, districts and voter groups who have a high chance of being pivotal. A considerable empirical literature documents how inequality in political influence produces inequality in public expenditures.[3] There is also evidence that voters who believe their participation to be pivotal turn out more likely (see, for example, Duffy and Tavits 2008) and these turnout rates affect policy (compare Mueller and Stratmann 2003 or Fumagalli and Narciso 2012).

The weighting implications of ensuring an equal say for all voters in a two-tier system were first formally considered by Lionel S. Penrose in 1946.[4] The institutional design of a

[2]Combining US voter figures with poll data, Gelman et al. (2012) estimated chances for a single vote being decisive in the 2008 presidential elections as about 1 in 60 million, but up to 1 in 10 million in some small and midsize states who were near the national median politically. Persuading 500 people in, e.g., New Mexico to change their votes would have provided a 1 in 6,000 chance of swinging the election ex ante.

[3]For instance, see Atlas et al. (1995) and Knight (2008) on US federal spending, Ansolabehere et al. (2002) on spending by US states, and Rodden (2002) for a study of the EU Council. On a related note, Strömberg (2008) has shown pivot considerations to be an important determinant for the allocation of campaign resources in US presidential elections.

[4]Informal investigations date back to anti-federalist writings by Luther Martin, a delegate from Maryland

successor to the League of Nations – today's UNO – was then being discussed. Penrose (1946) argued that the most intuitive solution to the democratic weight allocation problem, i.e., weights proportional to constituency sizes, ignores "elementary statistics of majority voting". Namely, if there are only two policy alternatives ('yes' or 'no') and all individual decisions are statistically *independent and equiprobable* then top-tier voting weights need to be such that the induced pivot probabilities of the delegates are proportional to the square root of represented population sizes. The corresponding suggestion is known as *Penrose's square root rule*.

This rule has provided a benchmark for a long list of applied studies on voting power in the US, EU, or IMF (see Felsenthal and Machover 2004; Grofman and Feld 2005; Fidrmuc et al. 2009; Leech and Leech 2009; Miller 2009, 2012, and the references therein). Politicians and diplomats may not care about Penrose's abstract statistical reasoning, but they have invoked his suggestion when it has fitted their interests.[5]

The special roles of square root and linear weight allocations have been confirmed, qualified, and disputed in numerous investigations of two-tier voting systems, both empirically (see Gelman et al. 2002, 2004) and theoretically. Besides the a priori influence of voters (Chamberlain and Rothschild 1981; Felsenthal and Machover 1998; Laruelle and Valenciano 2008b; Kaniovski 2008), utilitarian welfare maximization has played a particularly prominent role (e.g., Barberà and Jackson 2006; Beisbart and Bovens 2007; Laruelle and Valenciano 2008b; Koriyama et al. 2013). Moreover, avoidance of majoritarian paradoxes such as in the US presidential elections of 2000 has featured as a desirable ideal (Felsenthal and Machover 1999; Kirsch 2007; Feix et al. 2008).

This literature has made several departures from Penrose's original assumptions but has focused almost entirely on *binary* decisions.[6] Dichotomous options provide no scope

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to the Constitutional Convention in Philadelphia in 1787. See Riker (1986).

[5]A particularly notorious case involved the then Polish president and prime minister in the EU's negotiations of the Treaty of Lisbon. See, e.g., *The Economist* (2007, June 14th).

[6]We are aware of the following exceptions only: Laruelle and Valenciano (2008a) suggest a "neutral"

for negotiation and bargaining. This may be suited to political decisions between just two exogenously available candidates, or perhaps when deciding whether there is to be taxation, regulation, or aid. But it does not fit to competition between many policies and economic decisions on *levels*, such as rates of taxation or their progressiveness, intensity of regulation, aid's scale or means test requirements. We therefore analyze the equitable choice of voting weights for a richer set of alternatives. Though the relevant statistics are entirely different (and no longer so elementary) the design implications differ surprisingly little. Our analysis reinforces and corroborates an increasingly robust pattern in the literature: as originally argued by Penrose (1946), ex ante independent and identical voters require weight allocations based on the square roots of population sizes. But sufficient dissimilarity between constituencies renders most people's intuition correct – 'one person, one vote' calls for plain proportionality.

The remainder of the paper is organized as follows. In Section II, we spell out our stylized model of two-tier decision making and formalize the institutional design problem. Our main technical result, which characterizes pivot probabilities for a large number of constituencies, is presented in Section III. Its implications for equal representation are explored in Section IV. We conclude in Section V.

II. Model and Design Problem

A. Agents and Preferences

Consider a partition $\mathfrak{C}^m = \{C_1, \ldots, C_m\}$ of a large number n of voters into $m < n$ disjoint *constituencies* with $n_i = |C_i| > 0$ members each. The preferences of any voter $l \in \{1, \ldots, n\} =$

top-tier voting rule when policy alternatives give rise to a Nash bargaining problem. Le Breton et al. (2012) investigate fair voting weights in case of the division of a transferable surplus, i.e., for a simplex of policy alternatives. Maaser and Napel (2007, 2012, 2014) report Monte Carlo simulation results for influence-based, majoritarian, and welfarist objective functions in the median voter environment which we will here investigate analytically.

$\bigcup_i C_i$ are assumed to be single-peaked over a convex one-dimensional *policy space* $X \subseteq \mathbb{R}$, i.e., a finite or infinite real interval. Voter l's *ideal point* ν^l is conceived of as the realization of a continuous random variable. A given profile $(\nu^1, \ldots, \nu^n)$ of ideal points could reflect voter preferences in an abstract left–right spectrum or regarding a specific one-dimensional variable such as the location or scale of a public good, the level of a monetary transfer, an exemption threshold, etc.

We assume throughout our analysis that voter ideal points are a priori *identically distributed*. Moreover, it is assumed that ideal points are mutually *independent across constituencies*. We do, however, allow for a specific form of ideal points being correlated within each constituency. In particular, we conceive of the ideal point ν^l of any voter $l \in C_i$ as the sum

$$\nu^l = \mu_i + \epsilon^l \tag{1}$$

of a constituency-specific shock μ_i which has distribution H and a voter-specific shock ϵ^l with distribution G and continuous density g. Variables $\epsilon^1, \ldots, \epsilon^n$ and $\mu_1, \ldots, \mu_m$ are all mutually independent. The variance of G, $\sigma_G^2 < \infty$, can be interpreted as a measure of *heterogeneity within each constituency*, reflecting natural variation of political and economic preferences. If distribution H of μ_i is non-degenerate, it reflects a common attitude component of preferences within each constituency. H's variance $\sigma_H^2 < \infty$ is a straightforward measure of *heterogeneity across constituencies*: even though it is assumed that preferences in all constituencies vary between left–right, high tax–low tax, etc. in a similar manner, the locations of the respective ranges of opinion can differ between constituencies from an interim perspective.

The *i.i.d. case* of $\mu_i \equiv 0 = \sigma_H^2$ in which the ideal points of all voters are independent, also within constituencies, is an important benchmark. Still, constituencies differ a priori in nothing but size also if $\sigma_H^2 > 0$: ν^l and ν^k are correlated with the same coefficient $\sigma_H^2/(\sigma_H^2 + \sigma_G^2)$ whenever $l, k \in C_i$, for every constituency $C_i \in \mathfrak{C}^m$.

B. Two-Tier Median Voter Model

A collective decision $x^* \in X$ on the issue at hand is taken by an assembly of representatives $\mathcal{R}^m$ which consists of one representative from each of the m constituencies. Without committing to any particular procedure for within-constituency preference aggregation – such as bargaining, electoral competition, or a central mechanism – it will be assumed that preferences of C_i's representative coincide with those of its respective median voter, i.e., the location of the ideal point of representative i is

$$\lambda_i \equiv \text{median}\,\{\nu^l : l \in C_i\}. \tag{2}$$

This leaves aside agency problems and other reasons for why the preferences of a constituency's representative might not be congruent or at least sensitive to its median voter.[7]

In the top-tier assembly $\mathcal{R}^m$, constituency C_i has voting weight $w_i \geq 0$. Any coalition $S \subseteq \{1, \ldots, m\}$ of representatives which achieves a combined weight $\sum_{j \in S} w_j$ above

$$q^m \equiv \frac{1}{2} \sum_{j=1}^{m} w_j, \tag{3}$$

i.e., which has a *simple majority* of total weight, is winning and can pass proposals to implement some policy $x \in X$. This voting rule is denoted by $[q^m; w_1, \ldots, w_m]$.

Now consider the random permutation of $\{1, \ldots, m\}$ that makes $\lambda_{k:m}$ the k-th leftmost ideal point among the representatives for any realization of $\lambda_1, \ldots, \lambda_m$. That is, $\lambda_{k:m}$ is their k-th order statistic. We will disregard the zero probability events of two or more

[7]See, e.g., Gerber and Lewis (2004) for empirical evidence on how district median voters and partisan pressures jointly explain legislator preferences, and for a short discussion of the related theoretical literature. We remark that Theorem 1 below will *not* require identity (2) to hold.

constituencies having identical ideal points and define the random variable P by

$$P \equiv \min\Big\{j \in \{1,\dots,m\}\colon \sum_{k=1}^{j} w_{k:m} > q^m\Big\}. \tag{4}$$

The ideal point $\lambda_{P:m}$ of representative $P\colon m$ cannot be beaten by any alternative $x \in X$ in a pairwise vote, i.e., it is in the *core* of the voting game defined by ideal points $\lambda_1,\dots,\lambda_m$, weights $w_1,\dots,w_m$ and quota q^m. We assume that the policy x^* agreed by $\mathcal{R}^m$ lies in the core. Whenever that is single-valued, $\lambda_{P:m}$ actually beats every other alternative $x \in X$ and is the so-called *Condorcet winner* in $\mathcal{R}^m$. In order to avoid inessential case distinctions, we assume that $\mathcal{R}^m$ agrees on $\lambda_{P:m}$ also in the non-generic cases of the entire interval $[\lambda_{P-1:m}, \lambda_{P:m}]$ being majority-undominated, i.e., the collective choice equals[8]

$$x^* \equiv \lambda_{P:m}. \tag{5}$$

Representative $P\colon m$ will be referred to as either the *pivotal representative* or the *weighted median* of $\mathcal{R}^m$. Banks and Duggan (2000) and Cho and Duggan (2009) provide equilibrium analysis of non-cooperative bargaining which supports policy outcomes inside or close to the core. Note that for x^* determined in this way, no constituency's median voter has an incentive to choose a representative whose ideal point differs from her own one, that is, to misrepresent her preferences (cf. Moulin 1980; Nehring and Puppe 2007).

C. Influence and Equal Democratic Representation

Individuals differ only with respect to the sizes of their constituencies a priori; hence the voting weights which are allocated to their representatives should not create a disadvantage for members of large constituencies, nor for those of any other size. Our corresponding objective is to implement the influence aspect of the 'one person, one vote'

[8] A sufficient condition for the core to be single-valued is that the vector of weights satisfies $\sum_{j\in S} w_j \neq q^m$ for each $S \subseteq \{1,\dots,m\}$.

principle. More precisely, given a partition $\mathfrak{C}^m = \{C_1, \ldots, C_m\}$ of n voters into constituencies and distributions G and H which describe heterogeneity of individual preferences within and across constituencies, we would like to allocate voting weights $w_1, \ldots, w_m$ such that each voter a priori has *equal influence on the collective decision* $x^* \in X$.

There are two complementing ways of operationalizing a voter's influence on x^*. They extend the classical approach in the analysis of binary elections, where an individual l is influential if the election is tied without l's vote (or one vote away from a tie). In that case, voter l is decisive in two senses: (1) the election outcome coincides with l's vote and is *sensitive* to it, i.e., a change in l's vote would change the outcome; (2) given the decisions of all others, individual l's *turnout* matters, i.e., it makes a difference to the outcome whether l votes or not (at least with a positive probability which depends on tie-breaking). Direct analogues in our continuous world are that (1′) x^* coincides (approximately) with l's ideal point and idiosyncratic shifts of ν^l translate into shifts of x^*, i.e., $\partial x^*/\partial \nu^l > 0$; and (2′) whether l expresses her preferences or not, i.e., whether ideal point ν^l is incorporated into the local median, affects x^*'s location.

In contrast to the binary world, the two types of influence are associated with different events in our setup. That voter $l \in C_i$ is influential in the sensitivity sense requires that *l is C_i's median voter* if the population size n_i is odd. By our assumptions – ideal points ν^l and ν^k are identically distributed and at least conditionally independent if $l, k \in C_i$ – the probability of l being the local median is $1/n_i$, i.e., *inversely proportional* to constituency size. Conditioning on the intersection of this event and that of C_i's representative being pivotal at the top tier, we have $\partial x^*/\partial \nu^l = 1$. Since events $\{\nu^l = \lambda_i\}$ and $\{x^* = \lambda_i\}$ are independent,[9] we can quantify a priori influence of voter $l \in C_i$ as

$$\mathbf{E}\left[\frac{\partial x^*}{\partial \nu^l}\right] = \frac{\pi_i(\mathcal{R}^m)}{n_i} \tag{6}$$

[9]The first event only entails information about the identity of C_i's median, not its location.

where

$$\pi_i(\mathcal{R}^m) \equiv \Pr(P\colon m = i) \tag{7}$$

denotes the probability of C_i's representative being the assembly's weighted median. The same follows for an even population size.[10] Equal influence on collective decisions hence demands that $\pi_i(\mathcal{R}^m)$ is proportional to n_i.

Influence in the turnout sense does not require l to be a median voter in her constituency because *every* member of C_i affects the constituency's median position by participating. Dropping a voter who is to the left of λ_i from sample $\{\nu^l : l \in C_i\}$ would give rise to a new median position $\lambda_i' > \lambda_i$ on the right; dropping one on the right shifts λ_i left. (If the median voter herself fails to participate, λ_i is replaced by the midpoint $\lambda_i' \gtrless \lambda_i$ of its neighbors.) The probability of a given individual $l \in C_i$ having *some* influence in this sense – and thus a reason to vote – is $\pi_i(\mathcal{R}^m)$. However, the *extent* of influence (conditional on pivotality of C_i in $\mathcal{R}^m$) depends on constituency size. To see this, suppose all ideal points are independent and distributed uniformly on $[0, 1]$. Then the expected location of the k-th left-most position in C_i is $k/(n_i + 1)$. C_i's expected median position is $\mathbf{E}[\lambda_i] = 1/2$, and for an even n_i would be replaced by $\mathbf{E}[\lambda_i'] = (n_i/2+1)/(n_i+1) > 1/2$ if a left-wing voter dropped out. The shift's expected size is $\mathbf{E}[\lambda_i' - \lambda_i] = 1/[2(n_i + 1)]$, i.e., it is about halved if population is doubled.

The effects of adding (or deleting) an observation to a given sample are rigorously studied in mathematical statistics, in the context of robust estimation. There, the *influence function* of the median functional has been shown to be

$$\psi(\nu^l) = \frac{\operatorname{sign}(\nu^l - \mathrm{M})}{2f(\mathrm{M})} \tag{8}$$

[10]The median position λ_i then is the midpoint of the ideal points of C_i's two middlemost voters. The probability of l being one of them is $2/n_i$, and then $\partial x^*/\partial \nu^l = \frac{1}{2}$ if C_i's representative is pivotal. This yields $\mathbf{E}[\partial x^*/\partial \nu^l] = \pi_i(\mathcal{R}^m)/n_i$ again.

where f denotes the density of ν^l's distribution function F, and M is F's median. It can be used in order to write

$$\lambda_i = \mathrm{M} + \frac{1}{n_i} \sum_{l \in C_i} \psi(\nu^l) + R_i \tag{9}$$

with a residual term R_i which vanishes in probability as $n_i \to \infty$.[11] That is, we may – with only small error – conceive of C_i's median position as the result of starting at the theoretical median M and then doing n_i equidistant jumps of size $1/[2f(\mathrm{M}) \cdot n_i]$ to the right or left depending on whether $\nu^l > \mathrm{M}$ or $\nu^l < \mathrm{M}$. The effect of an individual voter l expressing her preferences or not on λ_i is thus inversely proportional to C_i's population size.[12] Providing all voters with the same influence from turning out therefore requires proportionality of probability $\pi_i(\mathcal{R}^m)$ to n_i.

It follows that irrespective of which type of individual influence we seek to equalize across constituencies, our institutional design objective consists of solving the following

Problem of Equal Democratic Representation:

Find a mapping from constituency sizes $n_1, \ldots, n_m$ to voting weights $w_1, \ldots, w_m$ for the representatives in $\mathcal{R}^m$ such that

$$\frac{\pi_i(\mathcal{R}^m)}{\pi_j(\mathcal{R}^m)} \approx \frac{n_i}{n_j} \text{ for all } i, j \in \{1, \ldots, m\}. \tag{10}$$

One might conjecture that we can simply use population sizes as voting weights, assuming w_i translates into $\pi_i(\mathcal{R}^m)$ linearly. This would be too quick on two grounds: the latter

[11]See, e.g., van der Vaart (1998, Example 20.5 and Corollary 21.5).

[12]Equation (9) derives from what statisticians call *von Mises calculus* (after the younger brother of the Austrian economist). It mimics Taylor approximation of a real function in the world of statistical functionals. Unfortunately, $R_i \in o_{F^{n_i}}(n_i^{-0.5})$; that is, the remainder term vanishes only at a square root rate in general. This means we cannot, strictly speaking, conclude directly from (9) that the expected size of a shift of λ_i due to dropping ν^l is proportional to n_i^{-1}. However, with a bit of effort, one can show that $\lim_{n_i \to \infty} 2(n_i + 1) f(\mathrm{M}) \cdot \Delta_{n_i}(f) = 1$, where $\Delta_{n_i}(f)$ is the expected change of the median caused by deleting one of n_i observations from the sample, given that all $\nu^l \in C_i$ are conditionally i.i.d. with continuous positive density f at M. A proof is available from the authors.

presumption can be unwarranted and, importantly, the distributions of representatives' ideal points affect pivot probabilities. The problem's solution will therefore depend on how constituencies' median preferences vary with their sizes, and how they interact with voting weights.[13]

D. Discussion

Before we start to investigate the stated design problem formally in Section III, two valid concerns should be addressed. A first objection to the pivotality-based condition (10) could be that a voter's associated indirect influence on outcomes is too small to worry about. We beg to differ because tiny numbers can matter. In particular, a level democratic playing field may be valued by the constituents as such: influence of a member of C_i should not be systematically larger or smaller than that of a member of C_j even if both are minuscule. Ratios are then more relevant than absolute levels. We would interpret public reaction to suggested re-weightings, for instance in the run-up to the Lisbon Treaty's reform of EU voting rules, along such non-instrumental lines. Moreover, small probabilities or policy shifts can matter also instrumentally. As discussed in detail by Edlin et al. (2007), it is an empirically plausible, rational explanation of *why* people vote that they care about the wider social benefits of policy (e.g., the entire government budget, not only what they personally get out of it or pay). If voters attend rallies or vote on a rainy election day against all odds because they perceive a sufficiently large stake, then pivotality has distributional consequences for personal welfare (as well as turnout incentives). Finally, though we derived condition (10) from democratic principles applied to individuals, it relates to the influence of *constituencies*. Chances of being decisive for an ultimate decision are key to

[13]Re-partitioning the population into constituencies of equal size – i.e., appropriate redistricting – is, of course, a possibility for altogether evading the considered problem. Our analysis is concerned with those cases where historical, geographical, cultural, and other reasons exogenously have defined a partition $\mathfrak{C}^m$ which cannot easily be changed. See Coate and Knight (2007) on socially optimal districting and Gul and Pesendorfer (2010) on strategic issues in redistricting.

a constituency's powers and have financial implications for it. They can be required to satisfy proportionality for reasons other than individual citizens' indirect influence.

With these arguments in mind, one might alternatively object that

$$\frac{\pi_i(\mathcal{R}^m)}{\pi_j(\mathcal{R}^m)} = \frac{n_i}{n_j} \ \textit{for all } i, j \in \{1, \ldots, m\} \tag{11}$$

should actually hold with equality, not just approximately. Unfortunately, due to the discrete nature of weighted voting, condition (11) cannot be satisfied by *any* weight vector $(w_1, \ldots, w_m)$ for most combinations of $\mathfrak{C}^m$, G, and H.[14] So the realistic task is either to minimize distance between $(n_1, \ldots, n_m)/n$ and the probability vector $(\pi_1(\mathcal{R}^m), \ldots, \pi_m(\mathcal{R}^m))$ induced by $w_1, \ldots, w_m$, or to find a way by which condition (11) is satisfied in an asymptotic sense – which corresponds to our notion of holding approximately. We follow the latter approach and, in particular, will not discuss here how one might solve the respective (non-trivial) discrete minimization problem for a specific partition $\mathfrak{C}^m$ and specific distributions G and H. Our ambition is to identify weighting rules which satisfy the 'one person, one vote' criterion approximately but generally, that is: they induce $\pi_i(\mathcal{R}^m)/\pi_j(\mathcal{R}^m) \approx n_i/n_j$ for any i, j for a large class of partitions $\mathfrak{C}^m$ and they require only qualitative information on voter heterogeneity.

To conclude this discussion, let us reiterate that the considered median voter model of two-tier decision making is admittedly a big simplification. Many collective decisions involve more than just a single dimension in which voter preferences differ. We ignore that voting might involve private information about some state variable (Feddersen and Pesendorfer 1996, 1997; Bouton and Castanheira 2012), and typical agency problems connected to imperfect monitoring and infrequent delegate elections. Empirical evidence highlights that a representative may take positions that differ significantly from his

[14] There are finitely many structurally different weighted voting games for any given m. Their number – related to *Dedekind's problem* in discrete mathematics – and the associated sets of feasible vectors $(\pi_1(\mathcal{R}^m), \ldots, \pi_m(\mathcal{R}^m))$ grow fast in m but include $(n_1, \ldots, n_m)/n$ only in special cases. See Kurz (2012).

district's median when voter preferences within that district are sufficiently heterogeneous (see, e.g., Gerber and Lewis 2004). Still, we take it that the best intuitions about fairness are captured by simplifying thought experiments of a veil of ignorance kind. The analysis of the described stylized world – no friction, particularly well-behaved preferences which are a priori identical for all – is useful in this way. It shows the limitations of and justifications for the simple intuition that weights should be proportional to the number of represented constituents, in a framework that goes beyond the binary world analyzed by Penrose and most others.

III. Pivot Probabilities for Many Constituencies

We now study how pivot probabilities $\pi_i(\mathcal{R}^m)$ in the assembly depend on voting weights and the ideal point distributions of delegates in general. We will then apply this knowledge to the problem of equal representation in Section IV.

Our perspective here is an asymptotic one, as in essentially all related literature since Penrose (1946). Studying the case when the number of constituencies is large has two benefits. It helps with the statistical analysis and, moreover, we avoid normative conundrums that can arise for a small number of constituencies. To see this second point, consider $m = 2$ with constituency C_1 twenty times more populous than C_2 and assume almost perfect preference correlation within constituencies. It is then very debatable whether $w_1 > w_2$ (dictatorship of C_1) or equal weights would be the 'fairer' top-tier voting rule. And the welfare loss, in utilitarian terms at least, of allowing delegate 2 to be pivotal could be enormous. Both unavoidable residual inequality as well as possible conflict between democratic fairness and utilitarian normative ideals decrease quickly as the number of independent voter groups rises.

Unfortunately, very few tangible results exist on the distribution of order statistics, like the median, from potentially *differently* distributed random variables (our delegate ideal points $\lambda_1, \ldots, \lambda_m$); almost nothing seems to be known about the distribution of a weighted median. It still turns out to be possible to characterize the probability $\pi_i(\mathcal{R}^m)$ of some delegate being the weighted median as $m \to \infty$. To this end, we conceive of $\mathcal{R}^1 \subset \mathcal{R}^2 \subset \mathcal{R}^3 \subset \ldots$ as a chain of assemblies with more and more members.

Any representative $i \in \mathbb{N}$ in the chain is endowed with a voting weight $w_i \geq 0$ and has a random ideal point λ_i with absolutely continuous distribution F_i. Some technical conditions will need to be imposed on weights and F_i's density. Yet, we can consider assemblies $\mathcal{R}^m$ with fairly arbitrary weighted voting rules $[q^m; w_1, \ldots, w_m]$ and independent ideal point distributions $F_1, \ldots, F_m$.[15] The obtained characterization thus may have applications that are unrelated to two-tier voting systems.

The considered sequences of weights $w_1, w_2, w_3, \ldots$ and ideal point distributions $F_1, F_2, F_3, \ldots$ are assumed to satisfy a weak form of replica structure. The reason is that otherwise our ratio of interest, $\pi_i(\mathcal{R}^m)/\pi_j(\mathcal{R}^m)$, need not converge.[16] We therefore require, first, that all representatives $i \in \mathbb{N}$ belong to one of an arbitrary but finite number of *representative types* $\theta \in \{1, \ldots, r\}$. All representatives are mutually independent but those of the same type have the same weight and ideal point distribution. So there exists a mapping $\tau\colon \mathbb{N} \to \{1, \ldots, r\}$ such that $\tau(j) = \theta$ implies that λ_j has distribution F_θ and $w_j = w_\theta$. If, second, each type θ maintains a *non-vanishing share*

$$\beta_\theta(m) \equiv |\{k \in \{1, \ldots, m\}\colon \tau(k) = \theta\}|/m \tag{12}$$

of representatives in $\mathcal{R}^m$ as $m \to \infty$, we call $\mathcal{R}^1 \subset \mathcal{R}^2 \subset \mathcal{R}^3 \subset \ldots$ a *regular chain*.

[15]A delegate might also represent the average ideal point inside a coalition government or reflect a member of a local oligarchy; weights might be unrelated to population sizes. Technically, for fixed $F_1, \ldots, F_m$, $\pi(\mathcal{R}^m)$ amounts to a specific *quasivalue* or *random order value*. See, e.g., Monderer and Samet (2002).

[16]This is illustrated by the sequence $\{\boldsymbol{w}^m\}_{m\in\mathbb{N}}$ with $\boldsymbol{w}^m = (1, 2, \ldots, 2) \in \mathbb{R}^m$. Representative 1 is either a null player with $\pi_1(\mathcal{R}^m) = 0$ or, supposing that the ideal point distributions $F_1, \ldots, F_m$ are identical, $\pi_i(\mathcal{R}^m) = \frac{1}{m}$ for *all* $i = 1, \ldots, m$ depending on whether m is odd or even. So $\pi_1(\mathcal{R}^m)/\pi_2(\mathcal{R}^m)$ alternates between 0 and 1. More complicated examples of non-convergence can be constructed by having $\{\boldsymbol{w}^m\}_{m\in\mathbb{N}}$ oscillate in a suitable fashion.

Regarding the ideal point distributions $F_1, F_2, F_3 \dots$ we require that they are *locally well-behaved* near their common median M: each associated density f_i is positive at M and varies at most quadratically, i.e., $f_i(\mathrm{M}) > 0$ and $|f_i(x) - f_i(\mathrm{M})| \leq c(x - \mathrm{M})^2$ for some $c \geq 0$ in a neighborhood of M.[17] With these restrictions, the following is shown to hold in the Appendix:

Theorem 1. *Consider a regular chain $\mathcal{R}^1 \subset \mathcal{R}^2 \subset \mathcal{R}^3 \subset \dots$ of assemblies. If the ideal point distribution F_θ has median* M *with locally well-behaved density f_θ for each representative type $\theta \in \{1, \dots, r\}$ then*

$$\lim_{m\to\infty} \frac{\pi_i(\mathcal{R}^m)}{\pi_j(\mathcal{R}^m)} = \frac{w_i f_i(\mathrm{M})}{w_j f_j(\mathrm{M})} \tag{13}$$

supposing that $w_j > 0$.

IV. Equal Representation

Theorem 1 allows to give a simple general answer to the problem of equal representation when many constituencies are involved. In particular, a comparison of equations (10) and (13) immediately yields that choosing

$$(w_1, \dots, w_m) \propto \left(\frac{n_1}{f_1(\mathrm{M})}, \dots, \frac{n_m}{f_m(\mathrm{M})}\right) \tag{14}$$

achieves approximately equal influence on collective decisions for all voters. The implicit presumption here is, of course, that the weights $w_1, \dots, w_m$ prescribed by (14) do not give rise to some of the problematic issues which regularity rules out in Theorem 1 (e.g., a dominant constituency that has a simple majority of weight).

[17] In the i.i.d. case, local well-behavedness of n_i individual ideal points – implied, e.g., by a symmetric C^2 density – is inherited by their median λ_i. A sufficient condition for well-behavedness in the non-i.i.d. case where ν^l has density $g * h\,(\mathrm{M}) > 0$ is that g and h are symmetric and C^1. Possibilities to relax the well-behavedness condition are discussed in an online appendix to the article.

A. Square Root Weights for Independent Constituents

When voter preferences are independent not only between but also within constituencies, Theorem 1 implies the following:

COROLLARY 1 (Square root rule). *If the ideal points of all voters are i.i.d. then choosing*

$$(w_1, \ldots, w_m) \propto \left(\sqrt{n_1}, \ldots, \sqrt{n_m} \right) \tag{15}$$

achieves equal representation asymptotically as $m \to \infty$.

The square roots of population sizes appear in relation (15) because λ_i is asymptotically (M, σ_i^2)-normally distributed, where $\sigma_i^2 = 1/([2g(\mathrm{M})]^2 \cdot n_i)$, if all ideal points are mutually independent (so $f \equiv g$). This is implied by equations (8)–(9) and the central limit theorem. We therefore have

$$f_i(\mathrm{M}) \approx \frac{1}{\sqrt{2\pi \cdot \frac{1}{[2g(\mathrm{M})]^2 \cdot n_i}}} = \frac{g(\mathrm{M})}{\sqrt{\pi/2}} \cdot \sqrt{n_i} > 0 \tag{16}$$

and just need to insert this into equation (14).[18]

In other words, in the i.i.d. case, the standard deviations of representatives' ideal points $\lambda_1, \ldots, \lambda_m$ are inversely proportional to the square roots of constituency sizes. Since λ_i's (approximately normal) density at the common expected median M is inversely proportional to its standard deviation, the constituency's probability to be the assembly's *unweighted* median is proportional to the square root of its size. This is illustrated in

[18] The error in approximation (16) vanishes quickly enough in order to conclude (15) from (14) also for moderately big population sizes $n_i \ll \infty$. In case of odd n_i, for instance, a well-known approximation of central binomial coefficients implies

$$\begin{aligned} f_i(\mathrm{M}) &= \binom{n_i - 1}{(n_i-1)/2} \cdot n_i G(\mathrm{M})^{(n_i-1)/2} [1 - G(\mathrm{M})]^{(n_i-1)/2} g(\mathrm{M}) \\ &= \left(1 + O\left(\tfrac{1}{n_i}\right)\right) \frac{4^{(n_i-1)/2}}{\sqrt{\pi(n_i-1)/2}} \cdot \frac{n_i}{2^{n_i-1}} g(\mathrm{M}) = \frac{g(\mathrm{M})}{\sqrt{\pi/2}} \cdot \sqrt{n_i} + O\left(\tfrac{1}{\sqrt{n_i}}\right). \end{aligned}$$

See, e.g., Arnold et al. (1992, p. 10) for a derivation of the exact density f_i in the first line.

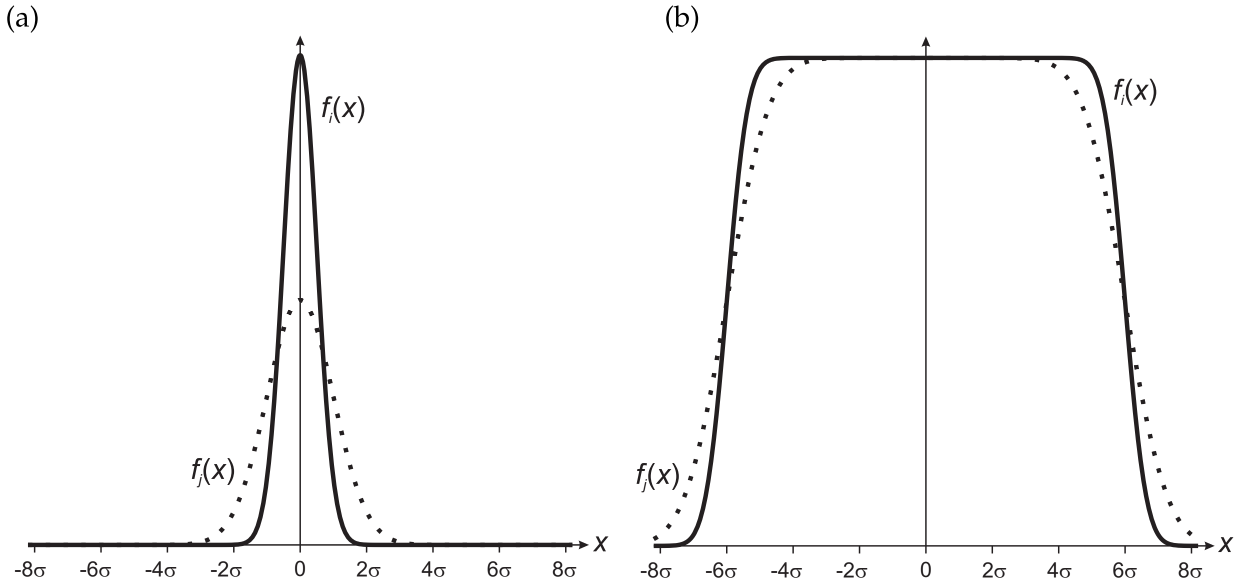


Figure 1: Densities of λ_i and λ_j when $n_i = 4n_j$ for (a) independent and (b) affiliated constituents

Figure 1(a). It shows the density functions of ideal points λ_i and λ_j for $n_i = 4n_j$, i.e., when constituency C_i is four times larger than constituency C_j. The representative of C_i has twice the chances to find himself in the political middle of his peers. Weights proportional to population sizes would hence give C_i more a priori influence than is due. Flat weights in turn would discriminate against C_i: its representative would be pivotal more often than his peers, but not proportionally more often. The balance is struck by relating to the square root of population sizes. Namely, a weight ratio of $w_i/w_j = \sqrt{4n_j}/\sqrt{n_j} = 2$ combined with $f_i(\mathrm{M})/f_j(\mathrm{M}) \approx 2$ yields $\pi_i(\mathcal{R}^m) \approx 4\pi_j(\mathcal{R}^m)$ as desired.

Difference to Penrose's Square Root Rule.—Corollary 1 echoes the finding of Penrose (1946) for the important i.i.d. benchmark case. One is tempted to suspect a deep common reason for this, rather than coincidence. However, the respective square root results come about very differently in both models. In the original binary setting, the non-linearity derives from the bottom tier: individual pivot probabilities match the probability of a tied local election, which asymptotically falls in $\sqrt{n_i}$ rather than n_i; square root weights ‘correct’ for this at the top tier.[19] In the interval case, by contrast, individual influences are inversely proportional to n_i at the bottom level. So voters from a large constituency suffer a proportional disadvantage. However, the distribution of their delegate’s ideal point is more concentrated around the political center – as opposed to a constant two-point distribution for all delegates in the binomial case. This creates an advantage at the top tier, which is absent in the binary setting. It rises proportionally to $\sqrt{n_i}$. So the net disadvantage for voters from a large constituency is the same in our and in Penrose’s models, but involves quite distinct bottom and top-tier components.

Analysis of cases in between the binary and interval ones, say, when individual ideal points are independent and identically distributed over $2 < s < \infty$ policy alternatives is very cumbersome. We have checked that voters’ bottom-tier disadvantage increases from square root ($s = 2$) to linear ($s = \infty$) at a speed which *differs* from that at which the indicated top-tier centrality advantage changes from non-existent to proportional to $\sqrt{n_i}$. So while the superposition of bottom and top-tier effects happens to imply a square root rule for i.i.d. voter preferences in both the binomial case and the interval case, the same does *not* in general apply to the multinomial case. This raises a first flag concerning policy recommendations which are justified by Corollary 1 and any binary analogues.

B. Proportional Weights for Affiliated Constituents

A second flag is prompted when we investigate the robustness of the square root rule regarding preference dependence. Adding a non-degenerate constituency-specific shock μ_i to voters’ idiosyncratic preference components ϵ^l creates positive correlation of the ideal points $\nu^l = \mu_i + \epsilon^l$ within a constituency. This reflects some polarization of preferences

[19]Note that Penrose’s square root rule does not directly relate to weights but to top-tier pivot probabilities, which in a binomial voting model equal the *(Penrose-)Banzhaf power index*. See Felsenthal and Machover (1998) or Laruelle and Valenciano (2008b) for good overviews.

along constituency lines which often seems natural.[20] It can here be measured simply by the ratio σ_H^2/σ_G^2.

Figure 1(b) illustrates the distribution of the median ideal point

$$\lambda_i = \text{median}\,\{\nu^l = \mu_i + \epsilon^l : l \in C_i\} = \mu_i + \text{median}\,\{\epsilon^l : l \in C_i\} \tag{17}$$

in constituency C_i when $\sigma_H^2/\sigma_G^2 > 0$. Density function f_i is virtually indistinguishable at M = 0 from λ_j's density function f_j – despite the fact that the population sample in C_i is four times that of C_j. Already small constituency-specific preference components suffice to induce $f_i(\mathrm{M})/f_j(\mathrm{M}) \approx 1$.[21] Then a weight ratio $w_i/w_j = 4$ which equals the population ratio is required in order to obtain $\pi_i(\mathcal{R}^m) \approx 4\pi_j(\mathcal{R}^m)$ and fair representation.

From an intuitive perspective, the effect of idiosyncratic noise on C_i's median opinion can be ignored for constituency sizes in the thousands or millions, unless the variance σ_H^2 of the constituency-specific shocks is several orders of magnitude smaller than σ_G^2. When both have similar sizes, the distribution of the median opinion and so delegate i's ideal point λ_i basically equals the distribution of the constituency-specific shocks μ_i around M. That distribution is the same for all constituencies; hence there is no centrality advantage for the delegate of a large constituency. Proportional pivot probabilities then require proportional weights.

Analytically, the switch from square roots to a *linear rule* is most easily seen for the case in which ϵ^l and μ_i are normal with zero means. Then, if ϵ^l has variance σ_G^2, the median of $\{\epsilon^l\}_{l\in C_i}$ is approximately normal with variance $\pi\sigma_G^2/(2n_i)$ (cf. Arnold et al. 1992,

[20]The basic features of *polarization* according to Esteban and Ray (1994, p. 824) are: (i) a high degree of homogeneity within groups, (ii) a high degree of heterogeneity across groups, and (iii) a small number of significantly sized groups.

[21] Specifically, we combine $\epsilon^l \sim \mathbf{N}(0, \sigma_G^2)$, where $\sigma_G^2 = (2n_j/\pi) \cdot \sigma^2$ for arbitrary $\sigma > 0$, with $\mu_i = \mu_j = 0$ in panel (a) and $\mu_i, \mu_j \sim \mathbf{U}[-6\sigma, 6\sigma]$ in panel (b). The latter entails a variance of $\sigma_H^2 = 12\sigma^2$. The polarization ratio depicted in panel (b), therefore, is $\sigma_H^2/\sigma_G^2 = 6\pi/n_j$. This would be puny if one thinks of n_j as referring to, e.g., the constituency size of a US state or EU member country.

Thm. 8.5.1). So when the constituency-specific preference component μ_i has variance σ_H^2, constituency C_i's aggregate ideal point λ_i is approximately normal with variance $\pi\sigma_G^2/(2n_i) + \sigma_H^2$. Considering the corresponding densities at M = 0 for representatives i and j yields

$$\frac{f_i(0)}{f_j(0)} = \left(\frac{\frac{\pi\sigma_G^2}{2n_i} + \sigma_H^2}{\frac{\pi\sigma_G^2}{2n_j} + \sigma_H^2} \right)^{-\frac{1}{2}}. \tag{18}$$

This equals $\sqrt{n_i}/\sqrt{n_i}$ at $\sigma_H^2 = 0$, as already observed. But the ratio quickly approaches 1 if $\sigma_H^2 > 0$ and $n_i, n_j \to \infty$. Theorem 1 then calls for proportionality of weights to population sizes, not their square roots. This does not hinge on assuming ϵ^l and μ_i to be normal, as one can check.[22] We can conclude:

Corollary 2 (Linear rule). *If individual ideal points are the sum of i.i.d. idiosyncratic components and i.i.d. constituency components with similar orders of magnitude then*

$$(w_1, \ldots, w_m) \propto (n_1, \ldots, n_m) \tag{19}$$

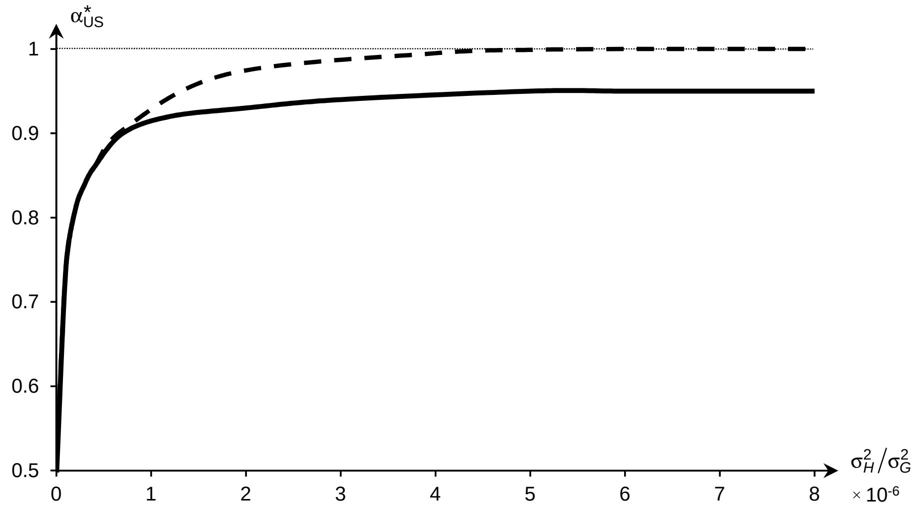
achieves equal representation asymptotically as $m \to \infty$.

C. Examples: the US and the European Union

Theorem 1 and its corollaries make asymptotic statements. They cannot, in general, characterize the best approximation of equal democratic representation for a given constituency configuration with fixed m. We remark, however, that numerical investigations have shown square root and linear weightings to play focal roles in achieving a level playing field already for partitions with ten to fifty constituencies (e.g., Maaser and

[22] In the discussion paper version, we prove that $\pi_i(\mathcal{R}^m)/\pi_j(\mathcal{R}^m)$ converges for any given m to the ratio of the Shapley values of C_i and C_j in the weighted majority game $[q^m; w_1, \ldots, w_m]$ as $[\pi\sigma_G^2/(2n_i)]/\sigma_H^2 \to 0$. This and the limit result of Neyman (1982) imply that Corollary 2 not only holds for the simple majority requirement q^m formalized by (3) but extends to *supermajority rules*. The latter is not true of Corollary 1.

(a)



(b)

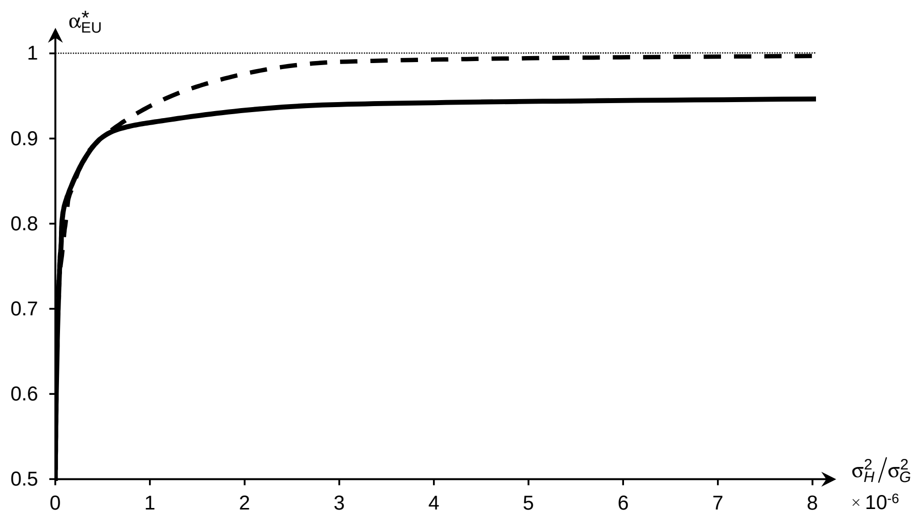


Figure 2: Best coefficient α for weight allocations $w_i \propto {n_i}^{\alpha}$ (solid line) and implicit weight assignments which induce Shapley value $\phi_i \propto {n_i}^{\alpha}$ (broken line) with (a) $n_1, \ldots, n_{51}$ defined by US population data and (b) $n_1, \ldots, n_{28}$ defined by EU28 population data

Napel 2007). Moreover, the key messages of Corollaries 1 and 2 are independent of the combinatorial aspect of weighted voting, which complicates matters for finite m.

These observations are illustrated in Figure 2. It considers (a) the US population's partition into 50 states and the District of Columbia, and (b) the European Union with currently 28 member states (EU28). The solid lines depict the (interpolated) coefficients α^*

which are optimal in the sense of minimizing $\|\cdot\|_1$-distance between individual influences and the democratic ideal of $(1/n, \ldots, 1/n) \in \mathbb{R}^n$ within the class of weighting rules

$$(w_1, \ldots, w_m) \propto (n_1{}^{\alpha}, \ldots, n_m{}^{\alpha}) \tag{20}$$

for $\alpha \in \{0, 0.01, \ldots, 1.99, 2\}$, as we assume increasing levels of polarization.[23] The equality of representation which is induced by $\alpha = 0.5$ is almost as equal as is feasible when $\sigma_H^2 = 0$. But Figure 2 shows the optimality of square root weights to break down quickly. Even small preference dissimilarity across the involved constituencies calls for weights that are close to proportional.

Relation to the Shapley Value.—One may wonder why the solid lines in Figure 2 do not reach $\alpha = 1$ as σ_H^2/σ_G^2 is raised, i.e., call for perfectly proportional weights? This owes to the indicated discrete nature of voting: for instance, weights of $(1, 1, 1)$, $(10\%, 45\%, 45\%)$ and $(49\%, 46\%, 5\%)$ all generate the same possibilities for three delegates to achieve a simple majority (by joining forces with any other delegate). This lumpiness means that pivot probabilities generally bear a non-linear relationship to voting weights, for any finite m. Corollary 2's suggestion of $\alpha = 1$ therefore cannot be expected to apply to the letter when, say, US or EU data is considered.

We note, however, that there is a long tradition of shedding light on the non-linearity between voters' weights and their possibilities to influence a collective decision – by means of *voting power indices*. The *Shapley value* (Shapley 1953) was established as such an index by Shapley and Shubik (1954).[24] Steps 2 and 3 in the proof of Theorem 1 identify a close connection between the value which is associated with a candidate weight assignment

[23] We consider $\epsilon^l \sim \mathbf{U}[-0.5, 0.5]$ and $\mu_i \sim \mathbf{N}(0, \sigma_H^2)$ with $0 \leq \sigma_H^2 \leq 10^{-6}$. Estimates of the pivot probabilities π_i which are induced by a given value of α are obtained by Monte Carlo simulation.

[24] The standard formula $\phi_i(v) \equiv \sum_{S \subseteq \{1,\ldots,m\}} |S|!\,(m - |S| - 1)!/m! \cdot [v(S \cup \{i\}) - v(S)]$ is simply applied to the characteristic function v with $v(S) = 1$ if $\sum_{i \in S} w_i > q^m$, and 0 otherwise. This is also referred to as the *Shapley-Shubik index* of rule $[q^m; w_1, \ldots, w_m]$. See Felsenthal and Machover (1998) or Laruelle and Valenciano (2008b).

$w_1, \ldots, w_m$ and the induced pivot probabilities. Exploiting this connection provides a route for getting closer to equality of representation than is possible with assignments $w_i \propto {n_i}^{\alpha}$, while still avoiding full optimization of weights on $\mathbb{R}_+^m$. In particular, there exist methods for finding voting weights such that a specified Shapley value $\tilde{\phi} \in \mathbb{R}^m$ is approximated well. One can thus consider *implicit* solutions to the problem of equal representation, i.e., suggestions of the form: select weights $w_1, \ldots, w_m$ such that the induced Shapley value is close to $({n_1}^{\alpha}, \ldots, {n_m}^{\alpha})$ for $\alpha \geq 0$. The best such suggestions – depending on the assumed degree of polarization among voters – are indicated by the broken lines in Figure 2. The associated $\|\cdot\|_1$-distances to $(1/n, \ldots, 1/n)$ become smaller than those when $w_i \propto {n_i}^{\alpha}$ by a factor of five as we raise σ_H^2/σ_G^2 for the US data and by a factor of 20 for the EU data, which involves fewer constituencies. Moreover, we (almost) obtain $\alpha = 1$ for high degrees of polarization in both panels.[25]

D. Discussion

For designing actual two-tier voting systems, it is obviously a contingent matter whether Corollary 2 for the case of affiliated constituents or Corollary 1 for the i.i.d. case provides better guidance. Some preference homogeneity within and dissimilarity across constituencies seems very plausible. It can arise as the result of a sorting process ('voting with one's feet') à la Tiebout (1956), be due to cultural uniformity fostered by proximity and local interaction (see Alesina and Spolaore 2003), or have other reasons. If constituencies correspond to entire nations, as for political and economic unions like the EU or the euro area, the members of a given constituency typically share more historical experience, industry, cultural and legal traditions, communication, etc. within constituencies than across. (That seems the key practical reason for why the issue of asymmetric constituency

[25]Targeting relative population sizes $(n_1, \ldots, n_m)/n$ with the Shapley value becomes *optimal* when σ_H^2 and σ_G^2 are of similar magnitude; see the technical remarks in fn. 22. Methods for moving from a given target value $\tilde{\phi}$ to suitable weights are described by Kurz (2012) or Kurz and Napel (2014).

sizes cannot trivially be resolved by redistricting.) This speaks clearly in favor of a linear rule. Still, the collective decisions which are taken by the top-tier assembly might be primarily about issues where opinions range over identical liberal–conservative or state–market spectrums in all constituencies. Moreover, there might be normative reasons outside the scope of our analysis for setting $\sigma_H^2 = 0$ when one designs a constitution. We therefore avoid specific recommendations here for, say, new voting rules in the Council of the European Union.[26] However, we warn that the i.i.d. presumption is considerably more 'knife-edge' and hence requires special motivation.

V. Concluding Remarks

We have extended the classical binary analysis of collective choice in two-tier systems to a median voter world with a continuum of alternatives. Our results broaden the basis for a priori assessments of voting weight arrangements, which differ widely in practice. Arguably, the informal balance of power between constituencies at the time of setting up a collective decision-making body matters more for the selection of a voting rule than normative arguments like ours. Still, such arguments have occasionally been taken up by practitioners (see, e.g., the Swedish diplomat Moberg 2010 on voting rules in the EU). In any case they clarify the premises behind competing claims that this or that system – plain proportionality, various forms of 'degressive proportionality', complete disregard for constituency sizes – is fairer than another.

Our analysis has operationalized the basic democratic principle of 'one person, one vote' as requiring proportionality between a constituency's size and the respective

[26]An interesting option, inspired by the call for flexible democratic mechanisms in other contexts (see Gersbach 2005, 2009), would be to specify different voting rules for different policy domains of an organization like the EU. In some areas, such as competition policy, small or unstable constituency differences may call for square root weights; fair decision making in other domains, such as agriculture or fisheries – with distinct shares of farmable land and sea access – could involve linear weights.

probability of getting its way. It turns out that this objective calls for approximate proportionality of voting weights to the square root of population sizes if individual voters' single-peaked preferences vary independently between and within constituencies. This finding is, however, not very robust. The more intuitive linear recommendation obtains if intra-constituency preference similarities are taken into account. Then, by default, weights ought to be proportional to population sizes. Adaptations which target a proportional Shapley value can improve this when the number of constituencies is small.

Although the equality of citizens' a priori influence on collective choices is a desirable ideal, it is of course not the only relevant benchmark. Pursuit of a welfarist design objective (find weights such that expected total utility of voters is maximized) and the majoritarian goal of minimizing the expected 'democracy gap' between the two-tier policy outcome and the outcome preferred by the population median come to mind. Simulations suggest that the policy implications of all these objectives are actually quite similar (see Maaser and Napel 2012, 2014).

Fairness, welfarist and majoritarian ideals are also linked theoretically: the hypothetical situation in which outcomes of a representative system perfectly imitate the predicted outcomes $x^{**} \equiv \text{median}\{\nu^l : l = 1, \ldots, n\}$ of direct democracy would necessarily involve proportionality of a constituency's pivot probability and its size for independent and identically distributed preferences. Welfare-maximizing voting weights would, similarly, try to bring the two-tier outcome x^* in congruence with the population's sample median if voters' utilities decrease linearly in distance to their respective ideals. So there are reasons to conjecture that our conclusions extend to interesting alternative desiderata. (If utility falls quadratically, and so total welfare is maximized by the sample mean, symmetry of the ideal point distributions would still make x^{**} an attractive target.)

Unfortunately, coincidence of the pivot probabilities in ideal majoritarian or utilitarian situations with the proportional pivot probabilities targeted by Corollaries 1 and 2 just provides a suggestive heuristic. Identity of x^* and x^{**} is, in general, unachievable. Pivot probabilities in the best feasible approximations may differ; so separate formal arguments are needed for each normative goal. Theorem 1 may help deriving them, as may the von Mises calculus which we invoked in order to operationalize individual influence (cf. fn. 12). The associated functional analytic methods are quite involved, and order statistics from non-identical distributions are unwieldy. We still hope that it will be possible to move beyond the prevailing binary focus on indirect collective choice, also for design objectives other than the one addressed here. It is an open challenge to obtain analogous results on the utilitarian and the majoritarian weights of nations.

Mathematical Appendix

Proof of Theorem 1

A. Overview

The major observation in the proof of Theorem 1 is that, as m grows large, the pivotal member of $\mathcal{R}^m$ is most likely found very close to the median M of distributions $F_1, \ldots, F_m$. Namely, the probability for the realized weighted median in $\mathcal{R}^m$ to fall outside a neighborhood of M turns out to approach zero at an exponential speed. One can therefore restrict attention to a neighborhood N_ε of M. In contrast to the more and more deterministic location of $\mathcal{R}^m$'s pivotal member,[27] the pivotal representative's identity remains a complicated, weight-dependent random variable even as $m \to \infty$. However, orderings of those representatives with ideal points inside N_ε become in good approximation conditionally equiprobable. Delegate i's conditional pivot probability, therefore, corresponds to i's Shapley value in 'subgames' which involve only the representatives j with realizations $\lambda_j \in N_\varepsilon$. It is then possible to apply the limit result proven by Neyman (1982) for the Shapley value and to exploit that the probability of condition $\{\lambda_i \in N_\varepsilon\}$ being true becomes proportional to λ_i's density at M when $\varepsilon \downarrow 0$.

The precise argument is structured into five steps. In *Step 1*, we define a particular neighborhood I_m of the expected location of the weighted median of $\lambda_1, \ldots, \lambda_m$. This *essential interval* I_m shrinks to {M} as $m \to \infty$. It is constructed such that the probabilities p_θ, $\overset{\sqsubset}{p}_\theta$, and $\overset{\sqsupset}{p}_\theta$ of a type-θ representative's ideal point falling inside I_m, inside I_m's left half, or inside I_m's right half, respectively, can suitably be bounded. Moreover, we decompose the deterministic total number $m_\theta = \beta_\theta(m) \cdot m$ of type-θ representatives in assembly $\mathcal{R}^m$

[27] If one took the assumptions of known preference distributions and an unbounded number m of constituencies literally, someone might suggest to dispense with voting and simply implement M. The limit consideration is, of course, only an analytical device. Numerical investigations, e.g., by Maaser and Napel (2007) confirm that the asymptotic findings are already a good guide for $10 < m < 50$.

into the random numbers ${}^{\triangleleft}\!k_\theta$, k_θ, and $k^{\triangleright}_\theta$ of delegates with ideal points to I_m's left, inside I_m, and to I_m's right. Knowing the respective vector $\mathbf{k} = ({}^{\triangleleft}\!k_1, k_1, k^{\triangleright}_1, \ldots, {}^{\triangleleft}\!k_r, k_r, k^{\triangleright}_r)$ is sufficient to determine whether the weighted median is located inside I_m or not.

In *Step 2*, it is established that the weighted median of $\lambda_1, \ldots, \lambda_m$ is located inside the essential interval I_m with a probability that exponentially approaches 1 as $m \to \infty$. As a corollary, the probability $\pi^\theta(\mathcal{R}^m)$ of the Condorcet winner having type θ converges to the corresponding *conditional* probability $\pi^\theta(\mathcal{R}^m|\mathcal{K})$ of a type-θ representative being pivotal where event $\mathcal{K}$ comprises all realizations of $\mathbf{k}$ such that $\mathcal{R}^m$'s weighted median lies inside I_m.

In *Step 3*, we show that the random orderings of the $k = \sum_{\theta \in \{1,\ldots,r\}} k_\theta$ representatives with ideal point realizations $\lambda_i \in I_m$ asymptotically become equiprobable as $m \to \infty$. It follows that, with a vanishing error, the respective conditional pivot probability $\pi^\theta(\mathcal{R}^m|\mathcal{K})$ equals the expected aggregate Shapley value of type-θ representatives in I_m.

In *Step 4*, the strong convergence result by Neyman (1982) is applied to our setting. Neyman's result implies that the aggregate Shapley value of type-θ representatives with ideal points in I_m converges to their respective aggregate voting weight in each considered weighted voting 'subgame' among the representatives with ideal points $\lambda_i \in I_m$.

Having established that $\pi^\theta(\mathcal{R}^m)$ is asymptotically proportional to the aggregate voting weight of all type-θ representatives with ideal points inside I_m, aggregate probabilities are broken down to individual representatives in the final *Step 5*. Note that ways to relax the conditions of the theorem are discussed in a separate online appendix.

B. Proof

Step 1: Essential interval I_m and vector k

We begin by identifying a neighborhood of M and a sufficiently great number m such that both the densities f_θ and the numbers of type-θ representatives in $\mathcal{R}^m$ can suitably be bounded. This leads to the definition of intervals I_m around M which later steps will focus on. Bounds for the probabilities of a type-θ representative's ideal point falling inside I_m, and more specifically into I_m's left or right halves, are provided in Lemma 1. The final part of *Step 1* introduces the vector $\mathbf{k}$ as a type-specific summary of how many ideal points are located to the left of I_m, inside I_m, and to its right.

First note that

$$0 < \underline{u} \equiv \min_{\theta' \in \{1,\dots,r\}} f_{\theta'}(\mathrm{M}) \leq f_\theta(\mathrm{M}) \leq \overline{u} \equiv \max_{\theta' \in \{1,\dots,r\}} f_{\theta'}(\mathrm{M}) \tag{21}$$

for every $\theta \in \{1, \dots, r\}$. Using the continuity of f_θ in a neighborhood $(\mathrm{M} - \varepsilon_1, \mathrm{M} + \varepsilon_1)$ of M, which is implied by $|f_\theta(x) - f_\theta(\mathrm{M})| \leq c(x - \mathrm{M})^2$, we can choose $0 < \varepsilon_2 \leq \varepsilon_1$ such that

$$\frac{5}{6} f_\theta(\mathrm{M}) \leq f_\theta(x) \leq \frac{7}{6} f_\theta(\mathrm{M}) \tag{22}$$

for all $x \in [\mathrm{M} - \varepsilon_2, \mathrm{M} + \varepsilon_2]$ and any specific $\theta \in \{1, \dots, r\}$. Inequality (21) can be used in order to obtain bounds

$$\frac{1}{2} \underline{u} \leq f_\theta(x) \leq 2\overline{u} \tag{23}$$

for all $x \in [\mathrm{M} - \varepsilon_2, \mathrm{M} + \varepsilon_2]$ and all $\theta \in \{1, \dots, r\}$ which do not depend on θ. The assumed regularity of $\mathcal{R}^1 \subset \mathcal{R}^2 \subset \mathcal{R}^3 \subset \dots$ entails the existence of some $m^0 \in \mathbb{N}$ such that $\beta_\theta(m) \geq \beta > 0$ for all $m \geq m^0$. So we can also choose $0 < \varepsilon_3 \leq \varepsilon_2$ such that

$$\beta_\theta(m) \geq \beta > 0 \tag{24}$$

for all $m \geq \frac{1}{{\varepsilon_3}^{8/3}}$ and all $\theta \in \{1, \dots, r\}$. And we can determine $0 < \varepsilon_4 \leq \varepsilon_3$ such that

$$24 < \underline{u}\beta \cdot (m\beta)^{\frac{1}{40}} \leq \underline{u}\beta m_\theta^{\frac{1}{40}} \tag{25}$$

for all $m \geq \frac{1}{{\varepsilon_4}^{8/3}}$, where $m_\theta \equiv \beta_\theta(m) \cdot m$.

Then define

$$\varepsilon(m) \equiv m^{-\frac{3}{8}} \tag{26}$$

and note that $\varepsilon(m) \leq \varepsilon_4$ iff $m \geq m^1 \equiv \frac{1}{\varepsilon_4{}^{8/3}} \geq m^0$. So, whenever we consider a sufficiently large number of representatives (specifically, $m \geq m^1$), inequalities (22)–(25) are satisfied. We refer to

$$I_m \equiv [\mathrm{M} - \varepsilon(m), \mathrm{M} + \varepsilon(m)] \tag{27}$$

as the *essential interval*. The probability of an ideal point of type θ to fall inside I_m is

$$p_\theta \equiv \int_{\mathrm{M}-\varepsilon(m)}^{\mathrm{M}+\varepsilon(m)} f_\theta(x)dx. \tag{28}$$

For realizations in the left and right halves of I_m we respectively obtain the probabilities

$$\overset{\sqsubset}{p}_\theta \equiv \int_{\mathrm{M}-\varepsilon(m)}^{\mathrm{M}} f_\theta(x)dx \quad \text{and} \quad \overset{\sqsupset}{p}_\theta \equiv \int_{\mathrm{M}}^{\mathrm{M}+\varepsilon(m)} f_\theta(x)dx, \tag{29}$$

with $\overset{\sqsubset}{p}_\theta + \overset{\sqsupset}{p}_\theta = p_\theta$.

Lemma 1. *For $m \geq m^1$ we have*

$$\frac{5}{3} f_\theta(\mathrm{M})\varepsilon(m) \leq \quad p_\theta \quad \leq \frac{7}{3} f_\theta(\mathrm{M})\varepsilon(m), \tag{30}$$

$$\frac{5}{6} f_\theta(\mathrm{M})\varepsilon(m) \leq \quad \overset{\sqsubset}{p}_\theta, \overset{\sqsupset}{p}_\theta \quad \leq \frac{7}{6} f_\theta(\mathrm{M})\varepsilon(m), \tag{31}$$

$$\underline{u}\beta m_\theta^{-\frac{3}{8}} \leq \quad p_\theta \quad \leq 4\overline{u} m_\theta^{-\frac{3}{8}}, \; \textit{and} \tag{32}$$

$$\frac{1}{2}\underline{u}\beta m_\theta^{-\frac{3}{8}} \leq \quad \overset{\sqsubset}{p}_\theta, \overset{\sqsupset}{p}_\theta \quad \leq 2\overline{u} m_\theta^{-\frac{3}{8}}. \tag{33}$$

Proof. The inequalities can be concluded from (22)–(24), $m_\theta = \beta_\theta m$, and $\beta < 1$. □

Now for any realization $\boldsymbol{\lambda}$ of the ideal points in assembly $\mathcal{R}^m$, let

$$k_\theta \equiv \#\{j\colon \tau(j) = \theta \text{ and } \lambda_j \in [\mathrm{M} - \varepsilon(m), \mathrm{M} + \varepsilon(m)]\} \tag{34}$$

denote the number of type-θ representatives with a policy position in the essential interval, i.e., no more than $\varepsilon(m)$ away from the expected sample median M. Analogously, let

$$\overleftarrow{k}_\theta \equiv \#\{j\colon \tau(j) = \theta \text{ and } \lambda_j \in (-\infty, \mathrm{M} - \varepsilon(m))\} \tag{35}$$

and

$$\overrightarrow{k}_\theta \equiv \#\{j\colon \tau(j) = \theta \text{ and } \lambda_j \in (\mathrm{M} + \varepsilon(m), \infty)\} \tag{36}$$

denote the random number of type-θ representatives to the left and to the right of I_m.

One can conceive of $\boldsymbol{\lambda}$-realizations as the results of two-part random experiments: in the first part, it is determined for each λ_j whether it is located to the right of I_m, to its left, or inside I_m, e.g., by drawing a vector $\boldsymbol{l} = (l_1, \ldots, l_m)$ of independent random variables where $l_i = 1$ (-1) indicates a realization of λ_i to the right (left) of I_m and $l_i = 0$ indicates $\lambda_i \in I_m$ (with probabilities $\frac{1}{2} - \overset{\sqsubset}{p}_\theta$, $\frac{1}{2} - \overset{\sqsupset}{p}_\theta$, and p_θ, respectively). This already fixes $\overleftarrow{k}_\theta$, k_θ, and $\overrightarrow{k}_\theta$ for each $\theta \in \{1, \ldots, r\}$ and is summarized by the vector

$$\mathbf{k} = (\overleftarrow{k}_1, k_1, \overrightarrow{k}_1, \ldots, \overleftarrow{k}_r, k_r, \overrightarrow{k}_r). \tag{37}$$

In the second part, the exact ideal point locations are drawn. It will turn out that those outside I_m can be ignored with vanishing error; and the k_θ type-θ ideal points inside have conditional densities $\hat{f}_\theta$ with

$$\hat{f}_\theta(x) \equiv \frac{f_\theta(x)}{p_\theta} \quad \text{for } x \in I_m. \tag{38}$$

Step 2: Type θ's aggregate pivot probability $\pi^\theta(\mathcal{R}^m)$ converges to the conditional probability $\pi^\theta(\mathcal{R}^m|\mathcal{K})$ of type θ being pivotal in I_m

We next appeal to *Hoeffding's inequality*[28] in order to obtain bounds on the probability that the shares of representatives ${}^{\triangleleft}k_\theta/m_\theta$, k_θ/m_θ, and $k^{\triangleright}_\theta/m_\theta$ with ideal points to the left, inside, or right of I_m deviate by more than a specified distance from their expectations. These bounds will imply that one can condition on the pivotal ideal point lying inside I_m in later steps of the proof with an exponentially decreasing error.

Hoeffding's inequality concerns the average $\overline{X} \equiv \frac{1}{n} \cdot \sum_{i=1}^{n} X_i$ of n independent bounded random variables $X_i \in [a_i, b_i]$ and guarantees

$$\Pr\left\{\left|\overline{X} - \mathbf{E}[\overline{X}]\right| > t\right\} \leq 2\exp\left(\frac{-2t^2n^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right). \tag{39}$$

Our specific construction will involve only random variables $X_i \in [0, 1]$, so that

$$\Pr\left\{\left|\overline{X} - \mathbf{E}[\overline{X}]\right| > t\right\} \leq 2\exp\left(-2t^2n\right). \tag{40}$$

We will put $n = m_\theta$ for a fixed $\theta \in \{1, \ldots, r\}$, so that $n \to \infty$ as $m \to \infty$, and choose $t = n^{-\frac{2}{5}}$, which implies $t(n) \ll \varepsilon(m)$. For this choice

$$\Pr\left\{\left|\overline{X} - \mathbf{E}[\overline{X}]\right| > n^{-\frac{2}{5}}\right\} \leq 2\exp\left(-2n^{\frac{1}{5}}\right), \tag{41}$$

i.e., the probability of "extreme realizations" exponentially goes to zero as $m \to \infty$ (and hence $n = m_\theta \to \infty$).

[28] See Hoeffding (1963, Theorem 2).

Lemma 2. *For each $\theta \in \{1,\dots,r\}$ we have:*

$$
\begin{array}{llcl}
\text{(I)} & \Pr\left\{\dfrac{\overset{\triangleleft}{k}_\theta}{m_\theta} \in \left[\dfrac{1}{2} - \overset{\sqsubset}{p}_\theta - m_\theta^{-\frac{2}{5}}, \dfrac{1}{2} - \overset{\sqsubset}{p}_\theta + m_\theta^{-\frac{2}{5}}\right]\right\} & \geq & 1 - 2\exp\left(-2m_\theta^{\frac{1}{5}}\right) \\
\text{(II)} & \Pr\left\{\dfrac{k_\theta}{m_\theta} \in \left[p_\theta - m_\theta^{-\frac{2}{5}}, p_\theta + m_\theta^{-\frac{2}{5}}\right]\right\} & \geq & 1 - 2\exp\left(-2m_\theta^{\frac{1}{5}}\right) \\
\text{(III)} & \Pr\left\{\dfrac{\overset{\triangleright}{k}_\theta}{m_\theta} \in \left[\dfrac{1}{2} - \overset{\sqsupset}{p}_\theta - m_\theta^{-\frac{2}{5}}, \dfrac{1}{2} - \overset{\sqsupset}{p}_\theta + m_\theta^{-\frac{2}{5}}\right]\right\} & \geq & 1 - 2\exp\left(-2m_\theta^{\frac{1}{5}}\right).
\end{array}
$$

Proof. Let $\theta \in \{1,\dots,r\}$ be arbitrary but fixed. For statement (I) we consider the $n = m_\theta$ indices $j_1 \dots, j_{m_\theta} \in \{1,\dots,m\}$ of type θ and denote by X_i the random variable which is 1 if the realization λ_{j_i} lies inside the interval $(-\infty, \mathrm{M} - \varepsilon(m))$ and zero otherwise. In the notation of Hoeffding's inequality we have $\overline{\mathrm{X}} = \overset{\triangleleft}{k}_\theta / m_\theta$. Since the probability that λ_{j_i} lies in the left half of I_m is given by $\overset{\sqsubset}{p}_\theta$ and $\int_{-\infty}^{\mathrm{M}} f_\theta(x)dx = \int_{\mathrm{M}}^{\infty} f_\theta(x)dx = \frac{1}{2}$, the probability that λ_{j_i} lies in the interval $(-\infty, \mathrm{M} - \varepsilon(m))$ is given by $\frac{1}{2} - \overset{\sqsubset}{p}_\theta$. Thus we have $\mathbf{E}[\overline{\mathrm{X}}] = \frac{1}{2} - \overset{\sqsubset}{p}_\theta$ and (41) implies (I). The statements (II) and (III) follow along the same lines (namely, by letting X_i be the characteristic function of intervals $[\mathrm{M} - \varepsilon(m), \mathrm{M} + \varepsilon(m)]$ and $(\mathrm{M} + \varepsilon(m), \infty)$, respectively). Note that ${m_\theta}^{-2/5} \ll \varepsilon(m) = m^{-3/8}$ for large m. □

We can use the bounds on p_θ in (32) and that $\beta m \leq m_\theta \leq m$ for $m \geq m^1 \geq m^0$ in order to conclude from (II) that for any given $\theta \in \{1,\dots,r\}$

$$
\underline{u}\beta^2 \varepsilon(m) \cdot m - m^{\frac{3}{5}} \leq k_\theta \leq 4\overline{u}\varepsilon(m) \cdot m + m^{\frac{3}{5}} \tag{42}
$$

with a probability of at least $1 - 2 \cdot \exp\left(-2{m_\theta}^{\frac{1}{5}}\right)$. A further implication of observations (I)–(III) is:

Lemma 3. *For $m \geq m^1$ the inequalities*

$$\begin{aligned} {}^{\triangleleft}k_\theta &< \frac{1}{2} m_\theta && (43) \\ k^{\triangleright}_\theta &< \frac{1}{2} m_\theta && (44) \\ {}^{\triangleleft}k_\theta + \frac{2}{3} k_\theta &> \frac{1}{2} m_\theta && (45) \\ k^{\triangleright}_\theta + \frac{2}{3} k_\theta &> \frac{1}{2} m_\theta && (46) \end{aligned}$$

are simultaneously satisfied for all $\theta \in \{1, \ldots, r\}$ *with a probability of at least* $1 - 6r \cdot \exp\left(-2(\beta m)^{\frac{1}{5}}\right)$.

Proof. The events considered in statements (I), (II), and (III) of Lemma 2 are realized for *all* $\theta \in \{1, \ldots, r\}$ with a joint probability of at least

$$\left(1 - 2\exp\left(-2(\beta m)^{\frac{1}{5}}\right)\right)^{3r} \geq 1 - 6r \exp\left(-2(\beta m)^{\frac{1}{5}}\right), \tag{47}$$

since $m_\theta \geq \beta m$ for $m \geq m^0$ and $(1-x)^k \geq (1-kx)$ is valid for all $x \in [0,1]$ and $k \in \mathbb{N}$. If $m \geq m^1$, we then have

$${}^{\triangleleft}k_\theta \leq \left(\frac{1}{2} - \overset{\sqsubset}{p}_\theta\right) \cdot m_\theta + m_\theta^{\frac{3}{5}} \leq \frac{m_\theta}{2} - \frac{u\beta m_\theta^{\frac{5}{8}}}{2} + m_\theta^{\frac{3}{5}} = \frac{m_\theta}{2} - m_\theta^{\frac{3}{5}} \underbrace{\left(\frac{u\beta m_\theta^{\frac{1}{40}}}{2} - 1\right)}_{>0} < \frac{1}{2} m_\theta \tag{48}$$

for any $\theta \in \{1, \ldots, r\}$. The first inequality follows directly from (I), the second inequality uses (33), and the final inequality follows from (25). Analogous inequalities pertain to $k^{\triangleright}_\theta$.

Moreover, we can conclude

$$
\begin{aligned}
{}^{\triangleleft}k_\theta + \frac{2}{3}k_\theta &\geq \left(\frac{1}{2} - \overset{\sqsubset}{p}_\theta\right) \cdot m_\theta - {m_\theta}^{\frac{3}{5}} + \frac{2p_\theta}{3}m_\theta - \frac{2}{3}{m_\theta}^{\frac{3}{5}} && (49)\\
&= \frac{m_\theta}{2} - \frac{5}{3}{m_\theta}^{\frac{3}{5}} + \left(\frac{2p_\theta}{3} - \overset{\sqsubset}{p}_\theta\right)m_\theta && (50)\\
&= \frac{m_\theta}{2} + \frac{5}{3}{m_\theta}^{\frac{3}{5}}\left(\frac{2\overset{\sqsupset}{p}_\theta}{5}{m_\theta}^{\frac{2}{5}} - \frac{\overset{\sqsubset}{p}_\theta}{5}{m_\theta}^{\frac{2}{5}} - 1\right) && (51)\\
&\geq \frac{m_\theta}{2} + \frac{5}{3}{m_\theta}^{\frac{3}{5}}\left(\frac{1}{10} \cdot \frac{3}{7}p_\theta \cdot {m_\theta}^{\frac{2}{5}} - 1\right) && (52)\\
&\geq \frac{m_\theta}{2} + \frac{5}{3}{m_\theta}^{\frac{3}{5}}\underbrace{\left(\frac{u\beta {m_\theta}^{\frac{1}{40}}}{24} - 1\right)}_{>0} > \frac{1}{2}m_\theta. && (53)
\end{aligned}
$$

The first inequality uses (I) and (II); the second one employs (30) and (31); the third applies (32); and the final one invokes (25). Analogous inequalities pertain to $k^{\triangleright}_\theta + \frac{2}{3}k_\theta$. □

Lemma 3 implies that the respective unweighted sample median among representatives of type θ is located within I_m for all $\theta \in \{1, \ldots, r\}$ with a probability that quickly approaches 1. The same must *a fortiori* be true for the pivotal assembly member, i.e., the weighted median among all representatives.

We collect in the set $\mathcal{K}$ all $\mathbf{k} = ({}^{\triangleleft}k_1, k_1, k^{\triangleright}_1, \ldots, {}^{\triangleleft}k_r, k_r, k^{\triangleright}_r)$ such that the events considered by Lemma 2, (I)–(III), are realized for *all* $\theta \in \{1, \ldots, r\}$. The inequalities in Lemma 3 then hold for any $\mathbf{k} \in \mathcal{K}$. We can decompose the probability $\pi^\theta(\mathcal{R}^m)$ of some type-θ representative being pivotal into conditional probabilities $\pi^\theta(\mathcal{R}^m|\mathcal{K})$ and $\pi^\theta(\mathcal{R}^m|\neg\mathcal{K})$ which respectively concern only λ-realizations where $\mathbf{k} \in \mathcal{K}$ and $\mathbf{k} \notin \mathcal{K}$. Then Lemma 3 implies

$$
\begin{aligned}
\pi^\theta(\mathcal{R}^m) &= \Pr\{\mathcal{K}\} \cdot \pi^\theta(\mathcal{R}^m|\mathcal{K}) + \Pr\{\neg\mathcal{K}\} \cdot \pi^\theta(\mathcal{R}^m|\neg\mathcal{K}) \\
&= \pi^\theta(\mathcal{R}^m|\mathcal{K}) + O(exp(-2m^{\frac{1}{5}})). && (54)
\end{aligned}
$$

Step 3: $\pi^\theta(\mathcal{R}^m|\mathcal{K})$ converges to the expectation of type θ's Shapley value inside I_m

Now condition on some $\mathbf{k} \in \mathcal{K}$ such that exactly $\sum_\theta k_\theta = k$ ideal points fall inside the essential interval, where k is asymptotically proportional to $\varepsilon(m) \cdot m = m^{\frac{5}{8}}$ by (42). Label them $1, \ldots, k$ for ease of notation and let $\varrho \in \mathcal{S}_k$ denote an arbitrary element of the space $\mathcal{S}_k$ of permutations which bijectively map $(1, \ldots, k)$ to some $(j_1, \ldots, j_k)$. The conditional probability for the event that the k ideal points located in I_m are ordered exactly as they are in ϱ by the second step of the experiment is

$$p(\varrho|\mathbf{k}) \equiv \int_{-\varepsilon(m)}^{\varepsilon(m)} \int_{x_{j_1}}^{\varepsilon(m)} \cdots \int_{x_{j_{k-1}}}^{\varepsilon(m)} \hat{f}_{j_1}(x_{j_1}) \ldots \hat{f}_{j_k}(x_{j_k})\, dx_{j_k} \ldots dx_{j_2} dx_{j_1}. \tag{55}$$

Lemma 4. *For all $m \geq m^1$, any $\mathbf{k} \in \mathcal{K}$ with $\sum_\theta k_\theta = k$ and permutation $\varrho \in \mathcal{S}_k$ we have*

$$p(\varrho|\mathbf{k}) = \frac{1}{k!} + \frac{1}{k!} \cdot O(m^{-\frac{1}{8}}). \tag{56}$$

Proof. The premise $|f_\theta(x) - f_\theta(\mathrm{M})| \leq c(x - \mathrm{M})^2$ for $x \in I_m$ permits us to choose $\delta \in O(\varepsilon(m)^2)$ with $\delta \leq \frac{1}{2}$ such that

$$(1 - \delta) \cdot f_\theta(\mathrm{M}) \leq f_\theta(x) \leq (1 + \delta) \cdot f_\theta(\mathrm{M}) \tag{57}$$

and, equivalently,

$$(1 - \delta) \cdot \hat{f}_\theta(\mathrm{M}) \leq \hat{f}_\theta(x) \leq (1 + \delta) \cdot \hat{f}_\theta(\mathrm{M}) \tag{58}$$

for all types $1 \leq \theta \leq r$ and all $x \in I_m$. Integrating (57) on I_m yields

$$2\varepsilon(m)(1 - \delta) \cdot f_\theta(\mathrm{M}) \leq p_\theta \leq 2\varepsilon(m)(1 + \delta) \cdot f_\theta(\mathrm{M}). \tag{59}$$

With these bounds we can conclude from $\hat{f}_\theta(\mathrm{M}) = \frac{f_\theta(\mathrm{M})}{p_\theta}$ that

$$\frac{1 - \delta}{2\varepsilon(m)} \leq \frac{1}{2\varepsilon(m)(1 + \delta)} \leq \hat{f}_\theta(\mathrm{M}) \leq \frac{1}{2\varepsilon(m)(1 - \delta)} \leq \frac{1 + 2\delta}{2\varepsilon(m)} \tag{60}$$

because $1/(1-\delta) \le 1+2\delta$.

Using $(1-\delta)^k \ge 1-k\delta$ and $(1+\delta)^k \le 1+2k\delta$ for $k\delta \le 1$,[29] and noting that the hypercube $[0,1]^k$ can be partitioned into $k!$ polytopes $\{x \in [0,1]^k \colon x_{j_1} \le x_{j_2} \le \ldots \le x_{j_k}\}$ with equal volume, inequality (58) yields

$$
\begin{aligned}
p(\varrho|\mathbf{k}) &\ge (1-\delta)^k \int_{-\varepsilon(m)}^{\varepsilon(m)} \int_{x_{j_1}}^{\varepsilon(m)} \cdots \int_{x_{j_{k-1}}}^{\varepsilon(m)} \hat{f}_{j_1}(\mathrm{M}) \ldots \hat{f}_{j_k}(\mathrm{M})\, dx_{j_k} \ldots dx_{j_2} dx_{j_1} && (61)\\
&= \frac{(1-\delta)^k}{k!} \cdot \hat{f}_{j_1}(\mathrm{M}) \ldots \hat{f}_{j_k}(\mathrm{M}) \int_{-\varepsilon(m)}^{\varepsilon(m)} \int_{-\varepsilon(m)}^{\varepsilon(m)} \cdots \int_{-\varepsilon(m)}^{\varepsilon(m)} 1\, dx_{j_k} \ldots dx_{j_2} dx_{j_1} && (62)\\
&= \frac{(1-\delta)^k}{k!} \cdot \hat{f}_{j_1}(\mathrm{M}) \ldots \hat{f}_{j_k}(\mathrm{M}) \cdot (2\varepsilon(m))^k && (63)\\
&\overset{(60)}{\ge} \frac{(1-\delta)^{2k}}{k!} \ge \frac{1-2k\delta}{k!} && (64)
\end{aligned}
$$

and, analogously,

$$
\begin{aligned}
p(\varrho|\mathbf{k}) &\le (1+\delta)^k \int_{-\varepsilon(m)}^{\varepsilon(m)} \int_{x_{j_1}}^{\varepsilon(m)} \cdots \int_{x_{j_{k-1}}}^{\varepsilon(m)} \hat{f}_{j_1}(\mathrm{M}) \ldots \hat{f}_{j_k}(\mathrm{M})\, dx_{j_k} \ldots dx_{j_2} dx_{j_1} && (65)\\
&= \frac{(1+\delta)^k}{k!} \cdot \hat{f}_{j_1}(\mathrm{M}) \ldots \hat{f}_{j_k}(\mathrm{M}) \cdot (2\varepsilon(m))^k && (66)\\
&\overset{(60)}{\le} \frac{(1+\delta)^k (1+2\delta)^k}{k!} \le \frac{(1+2\delta)^{2k}}{k!} \le \frac{1+8k\delta}{k!}. && (67)
\end{aligned}
$$

This implies

$$
\left| p(\varrho|\mathbf{k}) - \frac{1}{k!} \right| \le \frac{8k\delta}{k!}. \tag{68}
$$

Because $k \in O(m^{\frac{5}{8}})$ and $\delta \in O(m^{-\frac{6}{8}})$, the relative error $|p(\varrho|\mathbf{k}) - (k!)^{-1}| \big/ (k!)^{-1}$ tends to zero at

[29] The first statement is easily seen by induction on k. The second follows from

$$
(1+\delta)^k = \sum_{j=0}^{k} \binom{k}{j} \delta^j \le 1 + \sum_{j=1}^{k} \frac{1}{j!} \underbrace{(k\delta)^j}_{\le k\delta} \le 1 + k\delta \underbrace{\sum_{j=1}^{k} \tfrac{1}{j!}}_{\le e-1} \le 1 + 2k\delta.
$$

Since k is asymptotically proportional to $m^{\frac{5}{8}}$ and $\varepsilon(m)^2 = m^{-\frac{6}{8}}$ we can choose $\delta \in O(m^{-\frac{6}{8}})$ with $(k\delta)^j \le k\delta$ for $j \ge 1$ whenever m is large enough.

least as fast as $O(m^{-\frac{1}{8}})$. $\square$

So even though the probabilities of the orderings $\varrho \in \mathcal{S}_k$ of the k agents inside I_m differ depending on which specific ϱ is considered and what are the involved representative types (i.e., which $\mathbf{k}$ is considered), these differences vanish and all orderings become equiprobable as m gets large.

Type θ's conditional pivot probability can be written as

$$\pi^{\theta}(\mathcal{R}^m|\mathcal{K}) = \sum_{\mathbf{k}\in\mathcal{K}} P(\mathbf{k}) \cdot \Big\{ \sum_{\varrho\in\mathcal{S}_k\,:\,\psi(\mathbf{k},\varrho)=\theta} p(\varrho|\mathbf{k}) \Big\}, \tag{69}$$

where $P(\mathbf{k})$ denotes the probability of $\mathbf{k}$ conditional on event $\{\mathbf{k} \in \mathcal{K}\}$ and function $\psi\colon \mathcal{K} \times \mathcal{S}_k \to \{1, \ldots, r\}$ identifies the type θ' of the pivotal member in $\mathcal{R}^m$ when $\mathbf{k}$ describes how the representative types are divided between I_m and its left or right, and ϱ captures the ordering inside I_m. Lemma 4 approximates the probability of ordering ϱ conditional on $\mathbf{k}$ as $1/k!$, and one thus obtains

$$\pi^{\theta}(\mathcal{R}^m|\mathcal{K}) = \sum_{\mathbf{k}\in\mathcal{K}} P(\mathbf{k}) \cdot \phi_{\theta}(\mathbf{k}) + O(m^{-\frac{1}{8}}) \tag{70}$$

with

$$\phi_{\theta}(\mathbf{k}) = \sum_{\varrho\in\mathcal{S}_k\,:\,\psi(\mathbf{k},\varrho)=\theta} \frac{1}{k!}. \tag{71}$$

Because a constant factor $\frac{1}{k!}$ pertains to each ordering $\varrho \in \mathcal{S}_k$, $\phi_{\theta}(\mathbf{k})$ equals the probability that, as the weights $w_1, w_2, \ldots, w_k$ of the k representatives inside I_m are accumulated in *uniform* random order, the threshold $q(\mathbf{k}) \equiv q^m - \sum_{\theta\in\{1,\ldots,r\}} {}^{\triangleleft}k_{\theta} w_{\theta}$ is first reached by the weight of a type-θ representative. The term $\phi_{\theta}(\mathbf{k})$ is, therefore, simply the aggregated *Shapley value* of the type-θ representatives in the weighted voting game defined by quota $q(\mathbf{k})$ and weight vector $(w_1, w_2, \ldots, w_k)$. Equation (70) states that $\pi^{\theta}(\mathcal{R}^m|\mathcal{K})$ converges to the expectation of this Shapley value $\phi_{\theta}(\mathbf{k})$.

Step 4: Type θ's Shapley value ϕ_θ(k) converges to θ's relative weight in I_m

Condition $\mathbf{k} \in \mathcal{K}$ implies $\frac{1}{3} \cdot \sum_{\theta \in \{1,\dots,r\}} k_\theta w_\theta \leq q(\mathbf{k}) \leq \frac{2}{3} \cdot \sum_{\theta \in \{1,\dots,r\}} k_\theta w_\theta$ (see Lemma 3). And our premises guarantee that the relative weight of each individual representative in I_m shrinks to zero. The "Main Theorem*" in Neyman (1982), therefore, has the following corollary:

Lemma 5 (Neyman 1982). *Given that* $\mathbf{k} \in \mathcal{K}$,

$$\phi_\theta(\mathbf{k}) = \frac{k_\theta w_\theta}{\sum_{\theta'=1}^{r} k_{\theta'} w_{\theta'}} \cdot (1 + \mu(m)) \quad \textit{with} \quad \lim_{m \to \infty} |\mu(m)| = 0. \tag{72}$$

Proof. Neyman's theorem considers an infinite sequence of weighted voting games $[q^n; \boldsymbol{w}^n]$ with n voters whose individual relative weights w_i^n approach 0, and in which the relative quota q^n is bounded away from 0 and 100% (or at least $\lim_{n\to\infty} q^n/(\max_i w_i^n) = \infty$). Neyman establishes that[30]

$$\lim_{n\to\infty} |\phi_{T_n}(q^n; \boldsymbol{w}^n) - \sum_{i \in T_n} w_i^n| = 0 \tag{73}$$

holds for any sequence of voter subsets $T_n \subseteq \{1, \dots, n\}$, where $\phi_{T_n}(q^n; \boldsymbol{w}^n)$ denotes their aggregate Shapley value. (We here consider $q^n = q(\mathbf{k})/w_\Sigma$, $\boldsymbol{w}^n = (w_1, w_2, \dots, w_k)/w_\Sigma$ and $T_n = \{i \in N\colon \tau(i) = \theta\}$ for $N = \{1, \dots, k\}$ and $w_\Sigma = \sum_{i \in N} w_i$.[31])

It is trivial that (72) holds if $w_\theta = 0 = \phi_\theta(\mathbf{k})$. So we can assume $w_\theta > 0$, and because there is at least the proportion $\beta > 0$ of representatives from each type in I_m for large m, the aggregate relative weight of θ-type representatives in I_m is bounded away from 0, i.e.,[32]

$$\liminf_{m\to\infty} \frac{k_\theta w_\theta}{\sum_{\theta'=1}^{r} k_{\theta'} w_{\theta'}} > 0. \tag{74}$$

[30] We somewhat specialize his finding and adapt the notation.

[31] Our notation leaves some inessential technicalities implicit: $\mathcal{K}$ really refers to a family of such sets, parameterized by m; we implicitly consider a sequence of $\mathbf{k}$-vectors such that $n = k \to \infty$ as $m \to \infty$.

[32] The limit itself need not exist because our premises do not rule out that, e.g., m_θ is periodic in m.

Therefore, not only the absolute error $\tilde{\mu}(m)$ made in approximating $\phi_\theta(\mathbf{k}) = \phi_{T_n}(q^n; \boldsymbol{w}^n)$ by $\frac{k_\theta w_\theta}{\sum_{\theta'=1}^r k_{\theta'} w_{\theta'}}$ but also the relative error $\mu(m) \equiv \tilde{\mu}(m) / \frac{k_\theta w_\theta}{\sum_{\theta'=1}^r k_{\theta'} w_{\theta'}}$ must vanish as $m \to \infty$. □

Step 5: Attributing aggregate pivot probabilities to individual representatives

It remains to disaggregate the pivot probabilities $\pi^\theta(\mathcal{R}^m)$ and $\pi^{\theta'}(\mathcal{R}^m)$ of types θ and θ' to individual representatives i and j. The aggregate relative weight of type-θ representatives in the essential interval satisfies

$$\frac{k_\theta w_\theta}{\sum_{\theta'=1}^r k_{\theta'} w_{\theta'}} = \frac{\beta_\theta(m) m p_\theta w_\theta (1 + O(m^{-\frac{2}{5}}))}{\sum_{\theta'=1}^r \beta_{\theta'}(m) m p_{\theta'} w_{\theta'} (1 - O(m^{-\frac{2}{5}}))} = \frac{\beta_\theta(m) p_\theta w_\theta}{\sum_{\theta'=1}^r \beta_{\theta'}(m) p_{\theta'} w_{\theta'}} \left(1 + O(m^{-\frac{2}{5}})\right) \quad (75)$$

for any $\mathbf{k} \in \mathcal{K}$ (see (II) in Lemma 2).[33] Combining this with equations (54), (70) and (72) yields

$$\lim_{m\to\infty} \frac{\pi^\theta(\mathcal{R}^m)}{\pi^{\theta'}(\mathcal{R}^m)} = \lim_{m\to\infty} \frac{\beta_\theta(m) p_\theta w_\theta}{\beta_{\theta'}(m) p_{\theta'} w_{\theta'}} = \lim_{m\to\infty} \frac{\beta_\theta(m) f_\theta(\mathrm{M}) w_\theta}{\beta_{\theta'}(m) f_{\theta'}(\mathrm{M}) w_{\theta'}} \quad (76)$$

for arbitrary $\theta, \theta' \in \{1, \ldots, r\}$. Here, the final equality uses

$$\lim_{m\to\infty} \frac{p_\theta}{p_{\theta'}} = \lim_{m\to\infty} \frac{\int_{-\varepsilon(m)}^{\varepsilon(m)} f_\theta(x)dx}{\int_{-\varepsilon(m)}^{\varepsilon(m)} f_{\theta'}(x)dx} = \frac{f_\theta(\mathrm{M})}{f_{\theta'}(\mathrm{M})}, \quad (77)$$

which can be deduced from (59).

Our main claim then follows from noting that the $m_\theta = \beta_\theta(m) \cdot m$ representatives of type θ in assembly $\mathcal{R}^m$ are symmetric to each other and, therefore, must have identical pivot probabilities in $\mathcal{R}^m$. Hence

$$\lim_{m\to\infty} \frac{\pi_i(\mathcal{R}^m)}{\pi_j(\mathcal{R}^m)} = \lim_{m\to\infty} \frac{\pi^{\tau(i)}(\mathcal{R}^m)/\beta_{\tau(i)}(m)}{\pi^{\tau(j)}(\mathcal{R}^m)/\beta_{\tau(j)}(m)} = \frac{f_i(\mathrm{M}) w_i}{f_j(\mathrm{M}) w_j}. \quad (78)$$

[33] To see the second equality note that for $y \in (0, \frac{1}{2})$ we have $\frac{1}{1-y} = 1 + y + y^2 + \ldots \leq 1 + 2y = 1 + O(y)$. Similarly, $\frac{1}{1-y} \geq 1 + y = 1 + O(y)$ and so $\frac{1}{1-y} = 1 + O(y)$.

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