Large-volume open sets in normed spaces without integral distances

Sascha Kurz[*] Valery Mishkin[†]

Abstract

We study open sets $\mathcal{P}$ in normed spaces X attaining a large volume while avoiding pairs of points at integral distance. The proposed task is to find sharp inequalities for the maximum possible d-dimensional volume. This problem can be viewed as an opposite to known problems on point sets with pairwise integral or rational distances.

1 Introduction

For quite some time it was not known whether there exist seven points in the Euclidean plane, no three on a line, no four on a circle, with pairwise integral distances. Kreisel and Kurz [4] found such a set of size 7, but it is unknown if there exists one of size 8.[1] The hunt for those point sets was initiated by Ulam in 1945 by asking for a dense point set in the plane with pairwise rational distances.

Here we study a kind of opposite problem, recently considered by the authors for Euclidean spaces, see [5]: Given a normed space X, what is the maximum volume $f(X, n)$ of an open set $\mathcal{P} \subseteq X$ with n connected components without a pair of points at integral distance. We drop some technical assumptions for the normed spaces X and mostly consider the Euclidean spaces $\mathbb{E}^d$ or $\mathbb{R}^d$ equipped with a p-norm. In Theorem 5 we state an explicit formula for the Euclidean case $f(\mathbb{E}^d, n)$.

2 Basic notation and first observations

We assume that our normed space X admits a measure, which we denote by λ_X. By $\mathcal{B}_X$ we denote the open ball with diameter one in X, i.e. the set of points with distance smaller than $\frac{1}{2}$ from a given center. In the special case $X = (\mathbb{R}^d, \| \cdot \|_p)$ with $\| (x_1, \ldots, x_d) \|_X = \| (x_1, \ldots, x_d) \|_p :=$

[*]University of Bayreuth, D-95440 Bayreuth, Germany, sascha.kurz@uni-bayreuth.de

[†]York University, Toronto ON, M3J1P3 Canada, vmichkin@mathstat.yorku.ca

[1]We remark that, even earlier, Noll and Simmons, see http://www.isthe.com/chongo/tech/math/n-cluster, found those sets in 2006 with the additional property of having integral coordinates, so-called 7_2-clusters.

$\left(\sum_{i=1}^{d} |x_i|^p \right)^{\frac{1}{p}}$, where $p \in \mathbb{R}_{>0} \cup \{\infty\}$, we have

$$\lambda_X(\mathcal{B}_X) = \Gamma\left(\frac{1}{p} + 1\right)^d / \Gamma\left(\frac{d}{p} + 1\right),$$

where Γ denotes the famous Gamma function, i.e. the extension of the factorial function. In the Manhattan metric, i.e. $p = 1$, the volumes of the resulting cross-polytopes equal $\frac{1}{d!}$ and in the maximum norm, i.e. $p = \infty$, the volumes of the resulting hypercubes equal 1.

At first we observe that the diameter of a connected component $\mathcal{C}$ of a set $\mathcal{P} \subseteq X$ avoiding integral distances is at most 1.[2] Otherwise we can consider two points $u, v \in \mathcal{C}$ having a distance larger than 1 and conclude the existence of a point $w \in \mathcal{C}$ on the curve connecting u with v in $\mathcal{C}$ such that the distance between u and v is exactly 1.

Given two points $u, v \in X$, where $\|v - u\|_X = 1$, we may consider the line $\mathcal{L} := \{u + \alpha(v - u) \mid \alpha \in \mathbb{R}\} \subseteq X$. The restriction of X to $\mathcal{L}$ yields another, one-dimensional, normed space, where we can pose the same question.

We consider the map $\varphi : \mathcal{L} \to [0, 1)$, $p \mapsto \alpha \mod 1$ where $p = u + \alpha(v - u) \in \mathcal{L}$. If φ is not injective on $\mathcal{P} \cap \mathcal{L}$, the set $\mathcal{P}$ contains a pair of point an integral distance apart. Since the map φ is length preserving modulo 1, that is, $\|p_1 - p_2\|_X \mod 1 = |\varphi(p_1) - \varphi(p_2)|$, its restriction to the connected components of $\mathcal{P} \cap \mathcal{L}$ is length preserving. Thus, we conclude the necessary condition for an open set avoiding integral distances that the length of each intersection with a line is at most 1.

Having those two necessary conditions, i.e. the diameter of each connected component is at most 1 and the length of each line intersection is at most 1, at hand we define $l(X, n)$ as the maximum volume of open sets in X with n connected components, which satisfy the two necessary conditions. We thus have $f(X, n) \leq l(X, n) \leq n \cdot \lambda_X(\mathcal{B}_X)$ for all $n \in \mathbb{N}$.

In [5] the authors provided an example of a connected open set $U \subseteq \mathbb{R}^d$ such that the intersection of U with each line has a total length of at most 1 but the volume of U is unbounded.

Based on a simple averaging argument, any given upper bound on one of the two introduced maximum

[2]We remark that the maximum volume of a set with diameter 1 in X is at most $\lambda_X(\mathcal{B}_X)$, see e.g. [7].

volumes for n components induces an upper bound for $k \geq n$ components in the same normed space X:

Lemma 1 *For each $k \geq n$ we have,*

- $l(X, k) \leq \frac{k}{n} \cdot \Lambda$ *whenever* $l(X, n) \leq \Lambda$;

- $f(X, k) \leq \frac{k}{n} \cdot \Lambda$ *whenever* $f(X, n) \leq \Lambda$.

For lower bounds we consider the following construction. Given a small constant $0 < \varepsilon < \frac{1}{n}$, we arrange one open ball of diameter $1 - (n-1)\varepsilon$ and $n - 1$ open balls of diameter ε each, so that there centers are aligned on a line and that they are non-intersecting. Since everything fits into an open ball of diameter 1 there cannot be a pair of points at integral distance. As $\varepsilon \to 0$ the volume of the constructed set approaches $\lambda_X(\mathcal{B}_X)$, so that we have

$$\lambda_X(\mathcal{B}_X) \leq f(X, n) \leq l(X, n).$$

We then have $f(X, 1) \leq l(X, 1) = \lambda_X(\mathcal{B}_X)$. The map φ shows that equality is also attained for normed spaces X of dimension 1. So, in the following we consider sets consisting of at least two components and normed spaces of dimension at least two.

3 Two components

For one component the extremal example was the open ball of diameter 1. By choosing the line through the centers of two balls of diameter 1 we obtain a line intersection of total length two, so that this cannot happen in an integral distance avoiding set. The idea to circumvent this fact is to truncate the open balls in direction of the line connecting the centers so that both components have a width of almost $\frac{1}{2}$, see Figure 1 for the Euclidean plane $\mathbb{E}^2$.
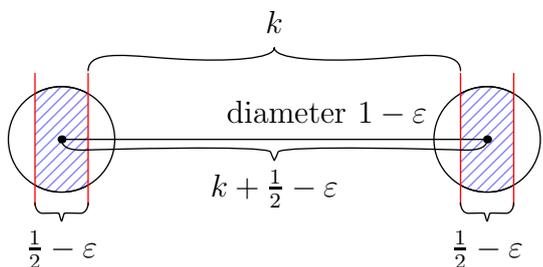

Figure 1: Truncated disks – a construction of 2 components in $\mathbb{E}^2$ without integral distances.

The volume V of a convex body $\mathcal{K} \subset \mathbb{E}^d$ with diameter D and minimal width ω is bounded above by

$$V \leq \lambda_{\mathbb{E}^{d-1}}(\mathcal{B}_{\mathbb{E}^{d-1}}) \cdot D^d \int_0^{\arcsin \frac{\omega}{D}} \cos^d \theta \, \mathrm{d}\theta, \qquad (1)$$

see e.g. [3, Theorem 1]. Equality holds iff $\mathcal{K}$ is the d-dimensional spherical symmetric slice with diameter D and minimal width ω. We can easily check that the maximum volume of two d-dimensional spherical symmetric slices with diameter 1 each and minimal widths ω_1 and ω_2, respectively, so that $\omega_1 + \omega_2 \leq 1$ is attained at $\omega_1 = \omega_2 = \frac{1}{2}$, independently from the dimension.

Motivated by this fact we generally define S_X to be a spherical symmetric slice with diameter 1 and width $\frac{1}{2}$, i.e. a truncated open ball. In the d-dimensional Euclidean case we have

$$\lambda_{\mathbb{E}^d}(S_{\mathbb{E}^d}) = \lambda_{\mathbb{E}^{d-1}}(\mathcal{B}_{\mathbb{E}^{d-1}}) \int_0^{\frac{\pi}{6}} \cos^d \theta \, \mathrm{d}\theta. \qquad (2)$$

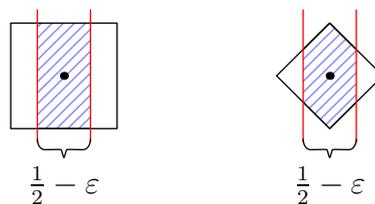The truncated disc in dimension $d = 2$ has an area of $\frac{\sqrt{3}}{8} + \frac{\pi}{12} \approx 0.4783$.

Figure 2: Spherical slices in dimension 2 for the maximum norm, $p = \infty$, and the Manhattan metric, $p = 1$.

On the left hand side of Figure 2 we have drawn the spherical slice, i.e. a truncated ball, in dimension 2 for the maximum norm, i.e. $p = \infty$. For general dimension d we have $\lambda_X(S_X) = \frac{1}{2}$. On the right hand side of Figure 2 we have drawn the spherical slice in dimension 2 for the Manhattan metric, i.e. $p = 1$. Here we have $\lambda_X(S_X) = \frac{1}{d!} - \frac{1}{2^{d-2}(d-1)!}$.

Since the line through the upper left *corner* of the left component and the lower right *corner* of the right component should have an intersection with the shaded region of total length at most 1, we consider truncated open balls of diameter $1 - \varepsilon$ and width $\frac{1}{2} - \varepsilon$, see Figure 1. For some special normed spaces X we can choose $\varepsilon > 0$ and move the centers of the two components sufficiently away from each other such that we can guarantee that no line intersection has a total length of more than 1.

Lemma 2 *For arbitrary dimension $d \geq 2$ and normed spaces $X = (\mathbb{R}^d, \|\cdot\|_p)$ with $1 < p < \infty$ we have $l(X, 2) \geq f(X, 2) \geq 2\lambda_X(S_X)$.*

Proof. For a given small $\varepsilon > 0$ place a truncated open ball of diameter $1 - \varepsilon$ and width $\frac{1}{2} - \varepsilon$ with its center at the origin and a second copy so that the two centers are at distance $k + \frac{1}{2} - \varepsilon$, see Figure 1 for $p = d = 2$. Since both components have

diameters smaller than 1 there cannot be a pair of points at integral distance within the same component. So let $a = (a_1, \ldots, a_d)$ be a point of the left and $b = (b_1, \ldots, b_d)$ a point of the right component, where we assume that the centers of the components are moved apart along the first coordinate axis. By construction the distance between a and b is at least k. Since $|a_i - b_i| < 1 - \varepsilon$ for all $2 \leq i \leq d$ and $|a_1 - b_1| < k + 1 - 2\varepsilon$ the distance between a and b is less than

$$\left((1 - \varepsilon)^p \cdot (d - 1) + (k + 1 - 2\varepsilon)^p \right)^{\frac{1}{p}}.$$

By choosing a sufficiently large integer k we can guarantee that this term is at most $k + 1$, so that there is no pair of points at integral distance. Finally, we consider the limit as $\varepsilon \to 0$. $\qquad\square$

We conjecture that the lower bound from Lemma 2 is sharp for all $p > 1$ and remark that this is true for the Euclidean case $p = 2$ by Inequality (1).

4 Relation to finite point sets with pairwise odd integral distances

In this section we restrict the connected components to open balls of diameter $\frac{1}{2}$. We remark that if an integral distance avoiding set contains an open ball of diameter $0 < D < 1$, which fits into one of its components, then the other components can contain open balls of diameter at most $1 - D$. One can easily check that the maximum volume of the entire set $\mathcal{P}$ is attained at $D = \frac{1}{2}$, at least for p-norms and dimensions $d \geq 2$. If the set $\mathcal{P}$, i.e. the collection of n open balls of diameter $\frac{1}{2}$, does not contain a pair of points at integral distance, then the mutual distances between centers of different balls have to be elements of $\mathbb{Z} + \frac{1}{2}$. Therefore dilating $\mathcal{P}$ by a factor of 2 yields the set $\mathcal{Q}$ of the centers of the n balls with pairwise odd integral distances.

However, for Euclidean spaces $\mathbb{E}^d$, it is known, see [2], that $|\mathcal{Q}| \leq d + 2$, where equality is possible if and only if $d \equiv 2 \pmod{16}$. It would be interesting to determine the maximum number of odd integral distances in other normed spaces.

5 Large-volume open sets with diameter and maximum length of line intersections at most one

Assuming that the construction using truncated open balls from Section 3 is best possible or, at the very least, competitive, we can try to arrange n copies of those S_X. Since we have to control that each line meets at most two components we cannot arrange the centers on a certain line. On the other hand, the cutting directions, i.e. the directions where we cut of the caps from the open balls, should be almost equal.

To meet both requirements, we arrange the centers of the components on a parabola, where each component has diameter $1 - \varepsilon$ and width $\frac{1}{2} - \varepsilon$, for a small constant $\varepsilon > 0$, see Figure 3 for an example in $\mathbb{E}^2$.
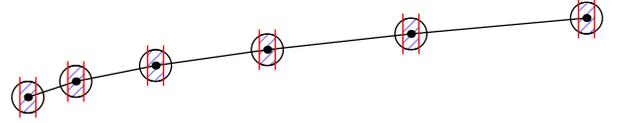

Figure 3: Truncated discs – arranged on a parabola.

Lemma 3 *For arbitrary dimension $d \geq 2$, $n \geq 2$, and normed spaces $X = (\mathbb{R}^d, \|\cdot\|_p)$ with $p > 1$ we have $l(X, n) \geq n\lambda_X(S_X)$.*

Proof. For a given small $\varepsilon > 0$ consider n truncated d-dimensional balls S_X of width $\frac{1}{2} - \varepsilon$, where the truncation is oriented in the direction of the y-axis, with centers located at $(i \cdot k, i \cdot k^2, 0, \ldots, 0)$ and diameter $1 - \varepsilon$ for $1 \leq i \leq n$, see Figure 3. For k large, there is no line intersecting three or more components. It remains to check that each line meeting two d-dimensional balls centered at C_1, C_2 has an intersection of length at most $\frac{1}{2}$ with each of the truncated balls. This can be done by performing a similar calculation as in the proof of Lemma 2. Again, we consider the limiting configuration as $\varepsilon \to 0$. $\qquad\square$

We conjecture that the lower bound in Lemma 3 is sharp.

6 Using results from Diophantine Approximation

In order to modify the construction from the previous section to obtain a lower bound for $f(X, n)$, we use results from Diophantine Approximation and:

Lemma 4 *Given an odd prime p, let $\alpha_j = \frac{\zeta_j - \zeta_{2p-j}}{i}$ for $1 \leq j \leq \frac{p-1}{2}$, where the ζ_j are $2p$th roots of unity. Then the α_j are irrational and linearly independent over $\mathbb{Q}$.*

A proof, based on a theorem of vanishing sums of roots of unity by Mann [6], is given in [5].

This result can be applied to construct sets avoiding integral distances as follows. We fix an odd prime p with $p \geq n$. For each integer $k \geq 2$ and each $\frac{1}{4} > \varepsilon > 0$ we consider a regular p-gon P with side lengths $2k \cdot \sin\left(\frac{\pi}{p}\right)$, i.e. with circumradius k. At n arbitrarily chosen vertices of the p-gon P we place the centers of d-dimensional open balls with diameter $1 - \varepsilon$. Since the diameter of each of the n components is less than 1 there is no pair of points at integral distance inside one of these n components. For each pair of centers c_1 and c_2, we cut off the corresponding two components

such that each component has a width of $\frac{1}{2} - \varepsilon$ in that direction, see Figure 4 for an example with $n = p = 5$.
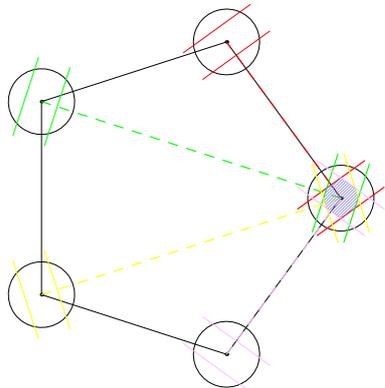

Figure 4: p-gon construction: Integral distance avoiding point set for $d = 2$ and $p = n = 5$.

Next we consider two points a and b from different components. We denote by α the distance of the centers of the corresponding components. From the triangle inequality we conclude

$$\alpha - \left(\frac{1 - 2\varepsilon}{2} \right) < \text{dist}(a, b) < \alpha + \left(\frac{1 - 2\varepsilon}{2} \right).$$

Since the occurring distances α are given by $2k \sin\left(\frac{j\pi}{p} \right)$ for $1 \leq j \leq \frac{p-1}{2}$ we look for a solution of the following system of inequalities

$$\left\{ 2k \cdot \sin\left(\frac{j\pi}{p} \right) - \frac{1}{2} + \varepsilon \right\} \leq 2\varepsilon \qquad (3)$$

with $k \in \mathbb{N}$, where $\{\beta\}$ denotes the positive fractional part of a real number β, i.e. there exists an integer l with $\beta = l + \{\beta\}$ and $0 \leq \{\beta\} < 1$. By Lemma 4 the factors $2 \sin\left(\frac{j\pi}{p} \right)$ are irrational and linearly independent over $\mathbb{Q}$, so by Weyl's Theorem [8] the systems admit a solution for all k. (Actually we only use the denseness result, which Weyl himself attributed to Kronecker.) We call the just described construction the p-gon construction.

These ingredients we provided in more detail in [5] enable us to establish the conjectured exact values of the function $f\left((\mathbb{E}^d, \|\cdot\|_2), n \right)$:

Theorem 5 *For all $n, d \geq 2$ we have*

$$f\left(\mathbb{E}^d, n \right) = n \cdot \lambda_{\mathbb{E}^d}(S_{\mathbb{E}^d}).$$

Proof. (Sketch)
For a given number n of components we choose an increasing sequence of primes $n < p_1 < p_2 < \dots$. For each $i = 1, 2, \dots$ we consider the p-gon construction with $p = p_i$ and n neighbored vertices. If the scaling

factor k tends to infinity the d-dimensional volume of the resulting sequence of integral distance avoiding sets tends to the volume of the following set: Let $\mathcal{P}_i$ arose as follows. Place an open ball of diameter 1 at n neighbored vertices of a regular p_i-gon with radius $2n^2$. Cut off caps in the direction of the lines connecting each pair of centers so that the components have a width of $\frac{1}{2}$ in that direction. In order to estimate the volume of $\mathcal{P}_i$ we consider another set $\mathcal{T}_i$, which arises as follows. Place an open ball of diameter $1 - \Delta_i$ at n neighbored vertices of a regular p_i-gon with radius $2n^2$. Cut off caps in the direction of the x-axis so that the components have a width of $\frac{1}{2} - \Delta_i$. One can suitably choose Δ_i so that $\mathcal{T}_i$ is contained in $\mathcal{P}_i$. Since the centers of the components of $\mathcal{P}_i$ tend to be aligned, as i increases, Δ_i tends to 0 as i approaches infinity. We thus have $\lim_{i \to \infty} \lambda_{\mathbb{E}^d}(\mathcal{T}_i) = n \cdot \lambda_{\mathbb{E}^d}(S_{\mathbb{E}^d})$. $\qquad \square$

7 Conclusion

We have proposed the question for the maximum volume of an open set $\mathcal{P}$ consisting of n components in an arbitrary normed space X avoiding integral distances. For the Euclidean plane those sets need to have upper density 0, see [1]. Theorem 5 proves a conjecture stated in [5] and some of the concepts have been transferred to more general spaces. Nevertheless, many problems remain unsolved and provoke further research.

References

[1] H. Furstenberg, Y. Katznelson, and B. Weiss, *Ergodic theory and configurations in sets of positive density*, Mathematics of Ramsey theory, Coll. Pap. Symp. Graph Theory, Prague/Czech., Algorithms Comb. 5, 184–198, 1990.

[2] R.L. Graham, B.L. Rothschild, and E.G. Straus, *Are there n + 2 points in E^n with odd integral distances?*, Amer. Math. Mon. **81** (1974), 21–25.

[3] M.A. Hernández Cifre, G. Salinas, and S. Segura Gomis, *Two optimization problems for convex bodies in the n-dimensional space*, Beiträge Algebra Geom. **45** (2004), no. 2, 549–555.

[4] T. Kreisel and S. Kurz, *There are integral heptagons, no three points on a line, no four on a circle*, Discrete Comput. Geom. **39** (2008), no. 4, 786–790.

[5] S. Kurz and V. Mishkin, *Open sets avoiding integral distances*, (submitted), http://arxiv.org/abs/1204.0403.

[6] H.B. Mann, *On linear relations between roots of unity*, Mathematika, Lond. **12** (1965), 107–117.

[7] M.S. Mel'nikov, *Dependence of volume and diameter of sets in an n-dimensional banach space*, Uspekhi Mat. Nauk **18** (1963), no. 4(112), 165–170.

[8] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352.