Regular Match-stick Graphs

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Abstract

A *match-stick graph* is a plane geometric graph in which every edge has length 1 and no two edges cross each other. It was conjectured that no 5-regular match-stick graph exists. In this paper we prove this conjecture.

1 Introduction

One of the possibly best known problems in combinatorial geometry asks how often the same distance can occur among n points in the plane, see e. g. [1]. Via scaling we can assume that the most frequent distance has length 1. Given any set P of points in the plane, we can define the so called unit-distance graph in the plane, connecting two elements of P by an edge if their distance is one. If we additionally require that the edges are non-crossing, then we obtain another class of geometrical and combinatorial objects:

Definition 1. A *match-stick graph* is a plane geometric graph in which every edge has length 1 and no two edges cross. (See for example the Harborth Graph in Figure 1.)

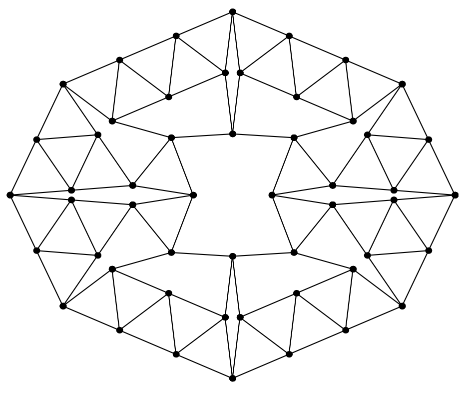

Figure 1: The Harborth Graph

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We call a match-stick graph r-regular if every vertex has degree r. In [4] the authors consider r-regular match-stick graphs with the minimum number $m(r)$ of vertices. Obviously we have $m(0) = 1$, $m(1) = 2$, and $m(2) = 3$, corresponding to a single vertex, a single edge, and a triangle, respectively.

The determination of $m(3)$ is an entertaining amusement. For degree $r = 4$ the exact determination of $m(4)$ is unsettled so far. The smallest known example is the so called Harborth graph, see e. g. [3], yielding $m(4) \leq 52$ (see Figure 1).

Due to the Eulerian polyhedron formula every finite planar graph contains a vertex of degree at most five so that we have $m(r) = \infty$ for $r \geq 6$. For $r = 5$ it was open whether $m(5)$ is finite, although it was conjectured (or believed) by several people that a finite 5-regular match-stick graph does not exist. In this paper we prove this conjecture.

We would like to mention the recent proof of the Higuchi conjecture [2] – a result of similar flavor where related techniques are applied.

2 5-regular match-stick graphs

Theorem 1. *No finite* 5*-regular match-stick graph does exist.*

Proof. Suppose to the contrary that there is such a graph M which we consider also as a planar map. Without loss of generality we assume that this graph is connected and denote by V the number of its vertices, by E the number of its edges, and by F the number of faces in the planar map M. By Euler's formula we have $V - E + F = 2$. For every $k \geq 3$ we denote by f_k the number of faces in M with precisely k edges.

We observe that $2E = \sum k f_k = 5V$ and $F = \sum f_k$. Therefore,

$$-6 = -3V + E + 2E - 3F = -3V + \frac{5}{2}V + \sum k f_k - 3\sum f_k = -\frac{1}{2}V + \sum (k-3) f_k. \qquad (1)$$

We begin by giving a charge of $-\frac{1}{2}$ to each vertex and by giving a charge of $k - 3$ to each face in M with precisely k edges. By (1) the total charge of all the vertices and faces is negative. We will reach a contradiction by redistributing the charge in such a way that eventually every vertex and every face will have a non-negative charge.

We redistribute the charge in the following very simple way. Consider a face T of M and a vertex x of T. Let α denote measure of the internal angle of T at x. Only if $\alpha > \frac{\pi}{3}$ we take a charge of $\min\left(\frac{1}{2}, \frac{3}{2\pi}\alpha - \frac{1}{2}\right)$ from T and move it to x (see Figure 2).

We now show that after the redistribution of charges every vertex and every face have a non-negative charge. Consider a vertex x. Let ℓ denote the number of internal angles at x that are greater than $\frac{\pi}{3}$. As the degree of x equals to 5 we must have $\ell > 0$. If due to one of these ℓ angles we transfered a charge of $\frac{1}{2}$ to x, then the charge at x is non-negative. Otherwise, note the the sum of these ℓ angles is at least $2\pi - (5-\ell)\frac{\pi}{3} = \frac{\pi}{3}(\ell+1)$. Hence the total charge transfered to x due to these angles is at least $\frac{3}{2\pi}\frac{\pi}{3}(\ell+1) - \frac{\ell}{2} = \frac{1}{2}$. Here again we conclude that the charge at x is non-negative.

Consider now a face T in M with $k \geq 3$ edges. Assume first that T is a bounded face. The initial charge of T is $k - 3 \geq 0$. Therefore, if the charge at T becomes negative this implies that

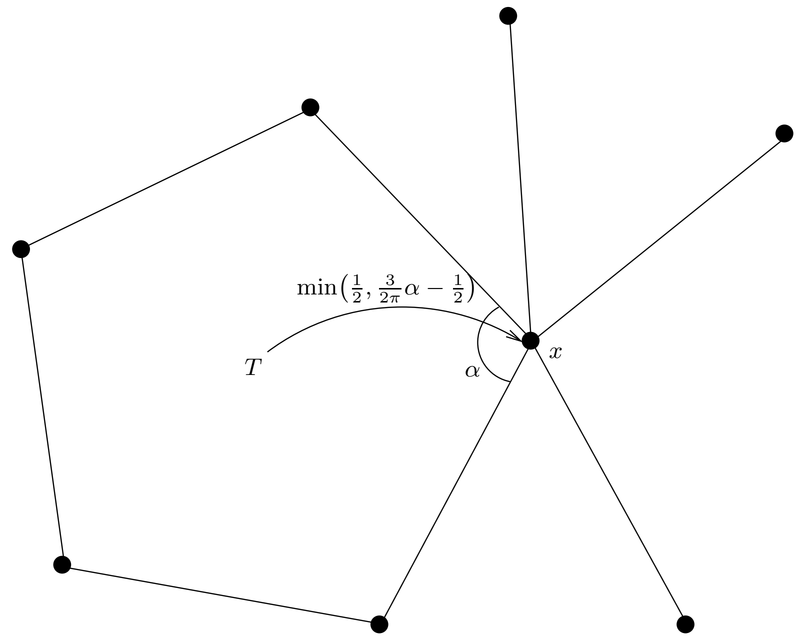

Figure 2: The distribution of a charge from a face T to its vertex x.

one of the internal angles of T is greater than $\frac{\pi}{3}$. In particular T cannot be a triangle and thus $k \geq 4$. If $k = 4$, then T is a rhombus. If only two internal angles of T are greater than $\frac{\pi}{3}$, then at most a total charge of 1 was deduced from the initial charge of T, leaving its charge non-negative. If all internal angles of T are greater than $\frac{\pi}{3}$, then the total charge deduced from T is at most $\frac{3}{2\pi} \cdot 2\pi - \frac{4}{2} = 1$, leaving the charge at T non-negative.

If $k = 5$ and the charge of T is negative after we redistribute the charges, then each of the internal angles of T must be greater than $\frac{\pi}{3}$ and as the sum of the internal angles of T is equal to 3π, the charge deduced from T amounts to at most $\frac{3}{2\pi} \cdot 3\pi - \frac{5}{2} = 2$, leaving the charge at T non-negative.

Finally if $k \geq 6$, then the charge deuced from T is at most $\frac{k}{2}$ leaving a charge of at least $k - 3 - \frac{k}{2} \geq 0$.

It is left to consider the unbounded face S of M. If the number of edges of S is at least 6, we are done as in the case of a bounded face. The cases where the unbounded face consist of at most 5 edges can be easily excluded. Another way to settle this issue is to observe that if S consists of at most 5 edges, then the total charge deduced from S is at most $\frac{5}{2}$ leaving the charge of S at least $-\frac{5}{2}$ (and in fact at least $-\frac{3}{2}$). We still obtain a contradiction as the sum of all charges should be equal to -6 while only the unbounded face may remain with a negative charge that is not smaller than $-\frac{5}{2}$. ∎

3 Concluding remarks

It is interesting to note that Theorem 1 is not true if we consider it on the sphere. A match-stick graph drawn on a sphere is a drawing of the vertices as points on the sphere and edges as great arcs connecting corresponding points, with the property that the lengths of all connecting arcs are equal and no two arcs cross. The example of the icosahedron shows that 5-regular match-stick graph may exist on a sphere (see Figure 3).

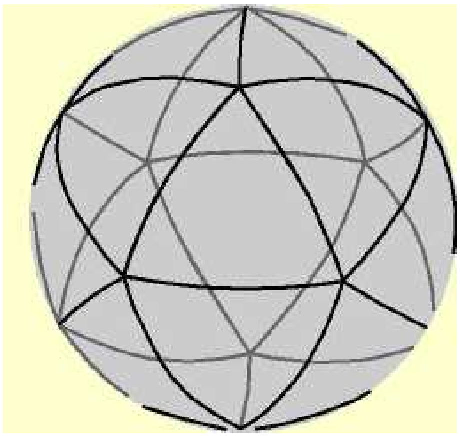

Figure 3: Icosahedron on a sphere (figure taken from www.grandunification.com).

References

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