

Counting polyominoes with minimum perimeter

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Abstract. The number of essentially different square polyominoes of order n and minimum perimeter $p(n)$ is enumerated.

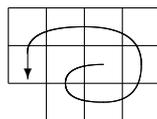
(In the published version: S. Kurz: Counting polyominoes with minimum perimeter, *Ars Combinatoria* Vol. 88 (2008), Pages 161-174, there were some problems for the case when the closed walk through the edge-to-edge neighboring squares of the perimeter contains a cut vertex. The example from Figure 3 can be treated if the unique square of degree four is counted twice. However, if the unique square of degree four is replaced by a chain of squares of degrees 3, 2, 2, \dots , 2, 3, the proofs of Lemma 1 and Lemma 2 collapse. This does not affect the main result, since for a large number of squares the corresponding graph of polyominoes with minimum perimeter cannot have cut vertices and it can be checked that for a small number of squares the main result is still valid. To avoid this inaccuracy, the current version circumvents the auxiliary results of Lemma 1 and Lemma 2.)

Keywords: polyominoes, enumeration

MSC: 05B50*, 05C30

1 Introduction

Suppose we are given n unit squares. What is the best way to arrange them side by side to gain the minimum perimeter $p(n)$? In [4] F. Harary and H. Harborth proved that $p(n) = 2\lceil 2\sqrt{n} \rceil$. They constructed an example where the cells grow up cell by cell like spirals for these extremal polyominoes (see Figure 1). In general, this is not



*sascha.kurz@uni-bayreuth.de Figure 1. Spiral construction.

the only possibility to reach the minimum perimeter. Thus the question arises to determine the number $e(n)$ of different square polyominoes of order n and with minimum perimeter $p(n)$ where we regard two polyominoes as equal if they can be mapped onto each other by translations, rotations, and reflections.

We will show that these extremal polyominoes can be obtained by deleting squares at the corners of rectangular polyominoes with the minimum perimeter $p(n)$ and with at least n squares. The process of deletion of squares ends if n squares remain forming a desired extremal polyomino. This process leads to an enumeration of the polyominoes with minimum perimeter $p(n)$.

Theorem 1. The number $e(n)$ of polyominoes with n squares and minimum perimeter $p(n)$ is given by

$$e(n) = \begin{cases} 1 & \text{if } n = s^2, \\ \left[-\frac{1}{2} + \frac{1}{2}\sqrt{1+4s-4t} \right] \sum_{c=0} r_{s-c-c^2-t} & \text{if } n = s^2 + t, \\ & 0 < t < s, \\ 1 & \text{if } n = s^2 + s, \\ q_{s+1-t} + \sum_{c=1}^{\lfloor \sqrt{s+1-t} \rfloor} r_{s+1-c^2-t} & \text{if } n = s^2 + s + t, \\ & 0 < t \leq s, \end{cases}$$

with $s = \lfloor \sqrt{n} \rfloor$, and with r_k, q_k being the coefficient of x^k in the following generating function $r(x)$ and $q(x)$, respectively. The two generating functions

$$s(x) = 1 + \sum_{k=1}^{\infty} x^{k^2} \prod_{j=1}^k \frac{1}{1 - x^{2j}}$$

and

$$a(x) = \prod_{j=1}^{\infty} \frac{1}{1 - x^j}$$

are used in the definition of

$$r(x) = \frac{1}{4} (a(x)^4 + 3a(x^2)^2)$$

and

$$q(x) = \frac{1}{8} (a(x)^4 + 3a(x^2)^2 + 2s(x)^2 a(x^2) + 2a(x^4)) .$$

Figure 2. $e(n)$ for $n \leq 100$.

The behavior of $e(n)$ is illustrated in Figure 2. It has a local maximum at $n = s^2 + 1$ and $n = s^2 + s + 1$ for $s \geq 1$. Then $e(n)$ decreases to $e(n) = 1$ at $n = s^2$ and $n = s^2 + s$. In the following we give lists of the values of $e(n)$ for $n \leq 144$ and of the two maximum cases $e(s^2 + 1)$ and $e(s^2 + s + 1)$ for $s \leq 49$,

$e(n) = 1, 1, 2, 1, 1, 1, 4, 2, 1, 6, 1, 1, 11, 4, 2, 1, 11, 6, 1, 1, 28, 11, 4, 2, 1, 35, 11, 6, 1, 1, 65, 28, 11, 4, 2, 1, 73, 35, 11, 6, 1, 1, 147, 65, 28, 11, 4, 2, 1, 182, 73, 35, 11, 6, 1, 1, 321, 147, 65, 28, 11, 4, 2, 1, 374, 182, 73, 35, 11, 6, 1, 1, 678, 321, 147, 65, 28, 11, 4, 2, 1, 816, 374, 182, 73, 35, 11, 6, 1, 1, 1382, 678, 321, 147, 65, 28, 11, 4, 2, 1, 1615, 816, 374, 182, 73, 35, 11, 6, 1, 1, 2738, 1382, 678, 321, 147,$

65, 28, 11, 4, 2, 1, 3244, 1615, 816, 374, 182, 73, 35, 11, 6, 1, 1, 5289, 2738, 1382, 678, 321, 147, 65, 28, 11, 4, 2, 1,

$e(s^2+1) = 1, 1, 6, 11, 35, 73, 182, 374, 816, 1615, 3244, 6160, 11678, 21353, 38742, 68541, 120082, 206448, 351386, 589237, 978626, 1605582, 2610694, 4201319, 6705559, 10607058, 16652362, 25937765, 40122446, 61629301, 94066442, 142668403, 215124896, 322514429, 480921808, 713356789, 1052884464, 1546475040, 2261006940, 3290837242, 4769203920, 6882855246, 9893497078, 14165630358, 20206501603, 28718344953, 40672085930, 57404156326, 80751193346,$

$e(s^2 + s + 1) = 2, 4, 11, 28, 65, 147, 321, 678, 1382, 2738, 5289, 9985, 18452, 33455, 59616, 104556, 180690, 308058, 518648, 863037, 1420480, 2314170, 3734063, 5970888, 9466452, 14887746, 23235296, 36000876, 55395893, 84680624, 128636339, 194239572, 291620864, 435422540, 646713658, 955680734, 1405394420, 2057063947, 2997341230, 4348440733, 6282115350, 9038897722, 12954509822, 18496005656, 26311093101, 37295254695, 52682844248, 74170401088, 104083151128.$

2 Proof of Theorem 1

The perimeter cannot be a minimum if the polyomino is disconnected or if it has holes. For connected polyominoes without holes the property of having the minimum perimeter is equivalent to the property of having the maximum number of common edges since an edge which does not belong to two squares is part of the perimeter. The maximum number of common edges $B(n)$, of a polyomino consisting of n unit squares, is determined in [4] to be $B(n) = 2n - \lceil 2\sqrt{n} \rceil$. Thus the perimeter of a polyomino consisting of n unit squares and having minimum perimeter is given by

$$2 \lceil 2\sqrt{n} \rceil. \quad (\star)$$

Lemma 1. For the maximum area $A(H)$ of a polyomino with perimeter H we have

$$A(H) = \begin{cases} \left(\frac{H}{4}\right)^2 & \text{if } |H| \equiv 0 \pmod{4}, \\ \left(\frac{H}{4}\right)^2 - \frac{1}{4} & \text{if } |H| \equiv 2 \pmod{4}. \end{cases}$$

Proof. W.l.o.g. we can consider a polyomino with minimum perimeter. Because the edges of the perimeter go either in the direction of the x -axis or the direction of the y -axis, the integer H has to be an even number. Consider the smallest rectangle surrounding a polyomino and denote the side lengths by a and b . Using the fact that the perimeter H of a polyomino is at least the perimeter of its smallest surrounding rectangle we conclude $H \geq 2a + 2b$. The maximum area of the rectangle with given perimeter is obtained if the integers a and b are as equal as possible. Thus $a = \lceil \frac{H}{4} \rceil$ and $b = \lfloor \frac{H}{4} \rfloor$. The product yields the asserted formula. \square

Now we give a strategy to construct all polyominoes with minimum perimeter. To this end we denote the degree of a square by the number of its edge-to-edge neighbors.

Lemma 2. Each polyomino with minimum perimeter $p(n)$ can be obtained by deleting squares, of degree 2, of a rectangular polyomino with perimeter $p(n)$ consisting of at least n squares.

Proof. Consider a polyomino P with minimum perimeter $H = p(n)$. Denote its smallest surrounding rectangle by R . If the perimeter of R is less than H then P does not have the minimum perimeter due to the fact that $m = B(n)$ is increasing. Thus H equals the perimeter of R and P can be obtained by deleting squares from a rectangular polyomino with perimeter $p(n)$ and with an area of at least n . Only squares of degree 2 can be deleted successively if the perimeter does not change. \square

For the following classes of n with $s = \lfloor \sqrt{n} \rfloor$ we now characterize all rectangles being appropriate for a deletion process to obtain P with minimum perimeter $p(n)$.

(i) $n = s^2$.

From Lemma 1 we know that the unique polyomino with minimum perimeter $p(n)$ is indeed the $s \times s$ square.

(ii) $n = s^2 + t, 0 < t < s$.

Since

$$s^2 < n < \left(s + \frac{1}{2}\right)^2 = s^2 + s + \frac{1}{4}$$

we conclude from Equation (\star) that the perimeter is given by $H = 4s + 2$. Denote the side lengths of the surrounding rectangle by a and b . With $2a + 2b = H = 4s + 2$ we let $a = s + 1 + c$ and $b = s - c$ with an integer $c \geq 0$. Since at least n squares are needed for the deletion process we have $ab \geq n$, yielding

$$0 \leq c \leq \left\lfloor -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4s - 4t} \right\rfloor.$$

(iii) $n = s^2 + s$.

The $s \times (s + 1)$ rectangle is the unique polyomino with minimum perimeter $p(n)$ due to Lemma 1.

(iv) $n = s^2 + s + t, 0 < t \leq s$.

Since

$$\left(s + \frac{1}{2}\right)^2 = s^2 + s + \frac{1}{4} < n < (s + 1)^2 = s^2 + 2s + 1$$

we conclude from Equation (\star) that the perimeter is given by $H = 4s + 4$. Again a and b denote the side lengths of the surrounding rectangle and we let $a = s + 1 + c$ and $b = s + 1 - c$ with an integer $c \geq 0$. The condition $ab \geq n$ now yields

$$0 \leq c \leq \left\lfloor \sqrt{1 + s - t} \right\rfloor.$$

We remark that the deletion process does not change the smallest surrounding rectangle since $ab - n < b$, that is the number of deleted squares is less than the number of squares of the smallest side of this rectangle. (Otherwise the perimeter would decrease.)

So far we have described those rectangles from which squares of degree 2 are removed. Now we examine the process of deleting squares from a rectangular polyomino. Squares of degree 2 can only be located in the corners of the polyomino. What shape has the set of deleted squares at any corner? There is a maximum square

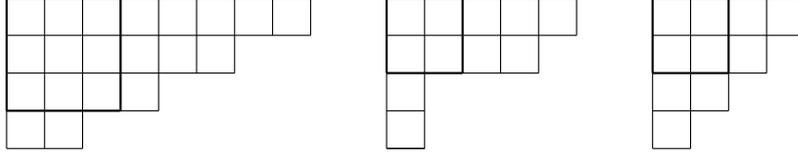


Figure 4. Shape of the deleted squares at the corners.

of squares at the corner, the so called “Durfee square”, together with squares in rows and columns of decreasing length from outside to the interior part of the polyomino. To count the different possibilities of the sets of deleted squares with respect to the number of the deleted squares we use the concept of a generating function $f(x) = \sum_{i=0}^{\infty} f_i x^i$. Here the coefficient f_i gives the number of different ways to use i squares. Since the rows and columns are ordered by their lengths they form Ferrer’s diagrams with generating function $\prod_{j=1}^{\infty} \frac{1}{1-x^j}$ each [2]. So the generating function for the sets of deleted squares in a single corner is given by

$$a(x) = \prod_{j=1}^{\infty} \frac{1}{1-x^j}.$$

Later we will also need the generating function $s(x)$ for the sets of deleted squares being symmetric with respect to the diagonal of the corner square. Since such a symmetric set of deleted squares consists of a square of k^2 squares and the two mirror images of a Ferrer’s diagrams with height or width at most k we get

$$s(x) = 1 + \sum_{k=1}^{\infty} x^{k^2} \prod_{j=1}^k \frac{1}{1-x^{2j}}.$$

We now consider the whole rectangle. Because of different sets of symmetry axes we distinguish between squares and rectangles. We define generating functions $q(x)$ and $r(x)$ so that the coefficient of x^k in $q(x)$ and $r(x)$ is the number of ways to remove k squares from all four corners of a square or a rectangle, respectively. We mention that the coefficient of x^k gives the desired number only if k is smaller than the small side of the rectangle.

Since we want to count polyominoes with minimum perimeter up to translation, rotation, and reflection, we have to factor out these symmetries. Here the general tool is the lemma of Cauchy-Frobenius, see e.g. [5]. We remark that we do not have to consider translations because we describe the polyominoes without coordinates.

Lemma (Cauchy-Frobenius, weighted form). Given a group action of a finite group G on a set S and a map $w : S \rightarrow R$ from S into a commutative ring R containing \mathbb{Q} as a subring. If w is constant on the orbits of G on S , then we have, for any transversal \mathcal{T} of the orbits:

$$\sum_{t \in \mathcal{T}} w(t) = \frac{1}{|G|} \sum_{g \in G} \sum_{s \in S_g} w(s)$$

where S_g denotes the elements of S being fixed by g , i.e.

$$S_g = \{s \in S \mid s = gs\}.$$

For G we take the symmetry group of a square or a rectangle, respectively, for S we take the sets of deleted squares on all 4 corners, and for the weight $w(s)$ we take x^k , where k is the number of squares in s . Here we will only describe in detail the application of this lemma for a determination of $q(x)$. We label the 4 corners of the square by 1, 2, 3, and 4, see Figure 5. In Table 1 we list the 8

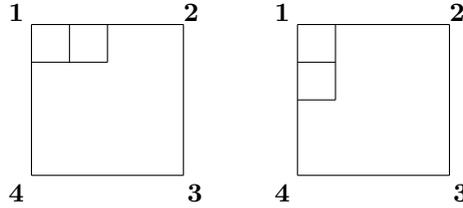


Figure 5.

permutations g of the symmetry group of a square, the dihedral group on 4 points, together with the corresponding generating functions for the sets S_g being fixed by g .

$(1)(2)(3)(4)$	$a(x)^4$
$(1, 2, 3, 4)$	$a(x^4)$
$(1, 3)(2, 4)$	$a(x^2)^2$
$(1, 4, 3, 2)$	$a(x^4)$
$(1, 2)(3, 4)$	$a(x^2)^2$
$(1, 4)(2, 3)$	$a(x^2)^2$
$(1, 3)(2)(4)$	$s(x)^2a(x^2)$
$(1)(2, 4)(3)$	$s(x)^2a(x^2)$

Table 1. Permutations of the symmetry group of a square together with the corresponding generating functions of S_g .

The generating function of the set of deleted squares on a corner is $a(x)$. If we consider the configurations being fixed by the identity element $(1)(2)(3)(4)$ we see that the sets of deleted squares at the 4 corners are independent and so $|S_{(1)(2)(3)(4)}| = a(x)^4$. In the case when $g = (1, 2, 3, 4)$ the sets of deleted squares have to be the same for all 4 corners and we have $|S_{(1,2,3,4)}| = a(x^4)$. For the double transposition $(1, 2)(3, 4)$ the sets of deleted squares at corners 1 and 2, and the sets of deleted squares at corners 3 and 4 have to be equal. Because the sets of deleted squares at corner points 1 and 3 are independent we get $|S_{(1,2)(3,4)}| = a(x^2)^2$. Next we consider $g = (1)(2, 4)(3)$. The sets of deleted squares at corners 2 and 4 have to be equal. If we apply g on the polyomino of the left hand side of Figure 5 we receive the polyomino

on the right hand side and we see that in general the sets of deleted squares at corners 1 and 3 have to be symmetric. Thus $|S_{(1)(2,4)(3)}| = s(x)^2 a(x^2)$. The other cases are left to the reader. Summing up and a division by 8 yields

$$q(x) = \frac{1}{8} (a(x)^4 + 3a(x^2)^2 + 2s(x)^2 a(x^2) + 2a(x^4)) .$$

For the symmetry group of a rectangle we analogously obtain

$$r(x) = \frac{1}{4} (a(x)^4 + 3a(x^2)^2) .$$

With the preceding characterization of rectangles being appropriate for a deletion process and the formulas for $a(x)$, $s(x)$, $q(x)$, and $r(x)$ we have the proof of Theorem 1 at hand.

We would like to close with the first entries of a complete list of polyominoes with minimum perimeter $p(n)$, see Figure 6.

3 Acknowledgments

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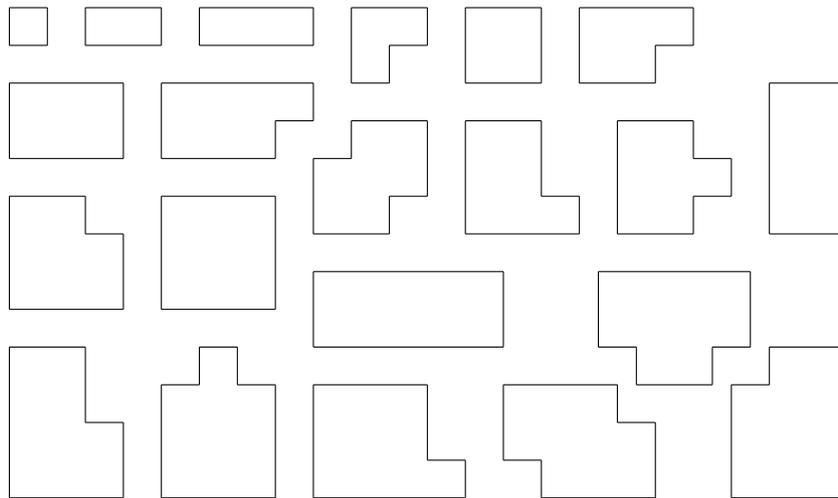


Figure 6. Polyominoes with minimum perimeter $p(n)$ for $n \leq 11$.

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