COMPUTING THE POWER DISTRIBUTION IN THE IMF

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Abstract. The International Monetary Fund is one of the largest international organizations using a weighted voting system. The weights of its 188 members are determined by a fixed amount of basic votes plus some extra votes for so-called Special Drawing Rights (SDR). On January 26, 2016, the conditions for the SDRs were increased at the 14th General Quota Review, which drastically changed the corresponding voting weights. However, since the share of voting weights in general is not equal to the influence, of a committee member on the committees overall decision, so-called power indices were introduced. So far the power distribution of the IMF was only computed by either approximation procedures or smaller games than then entire Board of Governors consisting of 188 members. We improve existing algorithms, based on dynamic programming, for the computation of power indices and provide the exact results for the IMF Board of Governors before and after the increase of voting weights. Tuned low-level details of the algorithms allow the repeated routine with sparse computational resources and can of course be applied to other large voting bodies. It turned out that the Banzhaf power shares are rather sensitive to changes of the quota.

Keywords: power indices, weighted voting games, International Monetary Fund, Shapley-Shubik index, Banzhaf index, empirical game theory
MSC: 91B12, 91A12

1. Introduction

The International Monetary Fund (IMF) was formed in 1944 at the Bretton Woods Conference. Currently this international organization consists of 188 countries as members. Its highest decision-making body, i.e., the *Board of Governors*, makes its decisions by weighted voting. The weights are composed of *basic votes*, which are equal for each member and sum up to 5.502 percent of the total votes, and one additional vote for each Special Drawing Right (SDR) of 100,000 of a member country's quota (the IMF term for the country's financial stake, c.f. [7]). On January 26, 2016, the conditions for the SDRs were increased at the Board Reform Amendment, which drastically changed the corresponding voting weights. In general the weight of a country can be a poor proxy for its influence in a weighted voting game.[1] To this end, so-called power indices were introduced in order to measure the *influence* or *power* of a committee member in a committee making its decisions via binary voting, i.e., each member can say "yes" or "no" to a given proposal. As the idea of power and influence is not defined unambiguously, several power indices were introduced in the literature. Arguably, the Shapley-Shubik and the Banzhaf index are two of the most frequently applied power indices. Unfortunately, the evaluation of such a power index is a computational hard problem in general.[2] And indeed, we are not aware of any paper, where either the Shapley-Shubik or the Banzhaf index of the IMF Board of Governors has been computed exactly. Approximation procedures were applied in [6, 7]. The *Executive Board* was, e.g., studied in [1]. In this paper we will compute the exact numerical values of both power indices for the IMF Board of Governors corresponding to voting weights slightly after and before the meeting on January 26, 2016. As the quota and voting shares will change as members pay their quota increases, see `https://www.imf.org/external/np/sec/memdir/members.aspx`, we list the used voting weights in tables 2-5.[3]

Algorithms for the efficient computation of power indices in voting games have been studied extensively in the literature. By looping over all 2^n subsets of players, the Shapley-Shubik index of a fixed player can be easily computed in $O(n \cdot 2^n)$ time. The straight-forward computation of the Banzhaf index of a fixed player can be performed in $O(n^2 \cdot 2^n)$ time. For weighted voting games these computation complexities were reduced to $O\left(n \cdot \sqrt{2}^n\right)$ and $O\left(n^2 \cdot \sqrt{2}^n\right)$ in [3], respectively. Assuming that all weights are integers and taking the sum of voting weights C into account, more

[1]Consider, e.g., a committee, where the weight shares are 49%, 49%, and 2%. For simple majority a least two out of the three committee members are needed in order to push through a proposal, i.e., the influences are equal contrary to the voting weights.

[2]To be more precise, the computation of the power indices treated in this paper is NP-hard in the sense of computational complexity theory. We give a brief justification at the end of Section 2.

[3]The voting weights were accessed at the official website `https://www.imf.org/external/np/sec/memdir/members.aspx`. The numbers were retrieved on February 17, 2016 and on July 27, 2015, respectively.

refined complexity results can be obtained. Several algorithms based on generating functions were implemented in `Mathematica`, see [10]. Those algorithms are fast if the subsets of players attain only few different weight sums. The number of different weight sums is clearly upper bounded by $C+1$. If almost all possible weight sums are attained, then one can use the related but conceptually easier concept of dynamic programming, see [9] for a survey.[4] With this, the Shapley-Shubik index of fixed player can be determined in $O(n^2q)$ time and $O(nq)$ space, where $q \le C$ denotes the *quota* of a weighted voting game. The Banzhaf index of a fixed player can be computed in $O(nq)$ time and $O(q)$ space. In [11] these complexity bounds are maintained for the computation of the respective power indices for all n players. We slightly improve upon these complexity bounds by replacing q by $\min(q, C-q+1)$,[5] provide an easy to understand description, and extend the analysis to further power indices. For practical efficiency we go into low-level details of the algorithms and discuss their impact on the running time for the IMF example.

The remaining part of this paper is structured as follows. In Section 2 we briefly introduce simple games as models for voting systems and some related notation. After introducing the defining equations for the power indices, we consider algorithms for their computation in Section 3. These are essentially based on counting the number of coalitions per weights and size by dynamic programming techniques. After stating our computational results in Subsection 3.4 we draw a conclusion in Section 4. The weights of the considered voting games and the resulting power distributions are outsourced into an appendix due to their large size.

2. Preliminaries

Let $N = \{1, \dots, n\}$ be the set of players. A *simple game* (on N) is a mapping $v : 2^N \to \{0, 1\}$ with $v(\emptyset) = 0$, $v(N) = 1$, and $v(S) \le v(T)$ for all $\emptyset \subseteq S \subseteq T \subseteq N$. A subset $S \subseteq N$ is called *coalition* and represents the set of "yes"-voters. A coalition S is called *winning* if $v(S) = 1$ and *losing* otherwise. A simple game v is *weighted* if there exist $q, w_1, \dots, w_n \in \mathbb{R}_{\ge 0}$ such that $v(S) = 1$ iff $w(S) \ge q$ for all $S \subseteq N$, where $w(S) := \sum_{i \in S} w_i$. The w_i are called *weights* (for player i) and q is called *quota*. We write $v = [q; w_1, \dots, w_n]$ and remark that weights and quota are far from being unique, so that we speak of a representation (q, w) for v. A representation with $q \in \mathbb{N}$, $w \in \mathbb{N}^n_{\ge 0}$ is called *integer representation*. It is well known that each weighted game admits an integer representation.[6] We speak of a *minimum sum integer representation* if the sum of weights is minimized within the class of all integer representations. Those representations need not to be unique in general if the number of players is not too small, see e.g. [4]. If $v(S \cup \{i\}) \ge v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$ we write $i \succeq j$, which defines a partial order. If this ordering is complete we call the simple game v *complete* and remark that all weighted games are complete. A player $i \in N$ is called a *null player* (in a simple game v), iff $v(S) = v(S \cup \{i\})$ for all $S \subseteq N \backslash \{i\}$. Two players $i, j \in n$ are called *equivalent*, denoted as $i \sim j$, if $i \succeq j$ and $j \succeq i$. If each winning coalition contains a certain player i, she is called *veto player*.

Next we briefly introduce the used power indices. The *Shapley-Shubik* index of player i is given by

$$\mathrm{SSI}_i(v) = \frac{1}{n!} \cdot \sum_{S \subseteq N \backslash \{i\}} |S|! \cdot (n - |S| - 1)! \cdot (v(S \cup \{i\}) - v(S)). \tag{1}$$

The *absolute Banzhaf index* of player i is given by

$$\mathrm{Bz}_i^{\mathrm{a}}(v) = \frac{1}{2^{n-1}} \cdot \sum_{S \subseteq N \backslash \{i\}} v(S \cup \{i\}) - v(S). \tag{2}$$

If we call a coalition $S \subseteq N \backslash \{i\}$ an *i-swing* if S is losing and $S \cup \{i\}$ winning, then $\mathrm{Bz}_i(v)$ is equal to the number of i-swings divided by the number of coalitions with(out) player i. Normalizing to sum 1, we obtain the (relative) *Banzhaf index* of player i:

$$\mathrm{Bz}_i(v) = \mathrm{Bz}_i^{\mathrm{a}}(v) / \sum_{j=1}^{n} \mathrm{Bz}_j^{\mathrm{a}}(v). \tag{3}$$

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[4] Actually, the only difference between the generating function and the dynamic programming approach is that the former utilizes the fast-access data structures for polynomials with few coefficients implemented in computer algebra systems. The generating function approach dates back at least to [8], where it was applied onto the electoral college.

[5] For the IMF Board of Governors we have $q = 0.85 \cdot C$, so that we obtain an acceleration of a factor of $0.85/0.15 \approx 5.67$. The memory requirements are reduced by the same factor.

[6] Let (q, w) be a representation of v, let α the maximum weight of losing coalition and β the minimum weight of a winning coalition. Increase the weights by at most $(\beta-\alpha)/2n > 0$ so that they become rational numbers. As quota chose an arbitrary rational number strictly between the new minimum weight of a winning coalition and the new maximum weight of a losing coalition. Multiplication with the common denominator yields an integer representation of v.

The two power indices have the property that they sum up to one and assign a value of zero to a player if and only if she is a null player. Since it is NP-hard to decide whether a player is a null player in a given weighted game, see e.g. [2], the computation of the used power indices is at least NP-hard. We remark that the equivalent players attain the same Shapley-Shubik or Banzhaf index.

3. Algorithms

Assume that we have a weighted game $v = [q; w]$ on n players in integer representation, where we set $C = \sum_{i=1}^{n} w_i$. As the complexity of our subsequent algorithms will depend on $\Delta := \min(q, C - q + 1)$ it would be beneficial to have a minimum sum integer representation at hand. However, it is not clear if minimizing the integer representation pays off for the computation of power indices, c.f. [5], where this is proposed as a promising strategy. So, here we propose to perform the following computationally cheap preprocessing steps at the very least. At first we reduce the weights that are larger then the quota by setting $q' = q$ and $w_i' = \min(q, w_i)$ for all $i \in N$. Next we guarantee that the weights are not too much larger than $C - q$. If $w_i > C - q$, then player i is a vetoer and we set $w_i' = C - q + 1$, $q' = q - w_i + w_i'$, and $w_j' = w_j$ for all $j \in N\backslash\{i\}$. Both operations can be performed in $O(n)$. The power indices used in this paper do not only assign zero power to all null players but are *null player preserving*, i.e., if v' arises from v by adding null player i, then we have $\mathcal{P}_j(v') = \mathcal{P}_j(v)$ for all $j \neq i$. Nevertheless, it is NP-hard to detect null players we can efficiently remove players with a zero weight, so that we can assume $1 \le w_i \le \Delta$ in the following, i.e., we have $C \ge n$.

In the following subsections we present the algorithmic details how to compute the power indices efficiently.

3.1. Counting coalitions per weight.

Let $c(x)$ denote the number of coalitions of a given weighted game v attaining weight x. By Algorithm 1 we can compute $c(x)$ for all $0 \le x \le q$ in $O(nq)$ time and $O(q+n)$ space, where we assume that we have precomputed the terms $\min\{q, \sum_{j=1}^{i} w_j\}$ for all $i \in N$.

Input: q, w, n
Output: $c(x)$ for $0 \le x \le q$
$c(0) \leftarrow 1$;
for $1 \le x \le q$ **do**
    $c(x) \leftarrow 0$;
end
for i *from* 1 *to* n **do**
    for x *from* $\min\{q, \sum_{j=1}^{i} w_j\}$ *to* w_i **do**
        $c(x) \leftarrow c(x) + c(x - w_i)$;
    end
end

Algorithm 1: Forward counting of coalitions per weight

Similarly we can compute the respective counts starting from weight C, see Algorithm 2 that needs $O(n \cdot (C - q + 1))$ time and $O(C - q + 1 + n)$ space.

Input: q, w, n
Output: $c(x)$ for $q \le x \le C$
$c(C) \leftarrow 1$;
for $q \le x \le C - 1$ **do**
    $c(x) \leftarrow 0$;
end
for i *from* 1 *to* n **do**
    for x *from* $\max\{q + w_i, C - \sum_{j=1}^{i-1} w_j\}$ *to* C **do**
        $c(x - w_i) \leftarrow c(x) + c(x - w_i)$;
    end
end

Algorithm 2: Backward counting of coalitions per weight

For the ease of notation we assume that the basic arithmetic operations for integers not too much larger than C can be performed in $O(1)$ time and space. However, the values stored in $c(x)$ can grow very quickly, i.e., we have $2^n \ge \max_{0 \le x \le C} c(x) \ge 2^n/(C + 1)$. So, we should count $\Theta(n)$ for each addition or subtraction. To avoid technical complications in the exposition and in order to be comparable with the related literature we also assume that all basic arithmetic operations for integers can be performed in constant time. From a practical point of view we have to deal with the corresponding problems nevertheless. In our application of the IMF we have $n = 188$, so that the values of $c(x)$ do not fit into the standard, simple data types on a 64-bit system. Since the overhead of a general-purpose arbitrary-precision arithmetic is quite large, we directly implement the most frequently used basic operations as follows. We choose different primes $p_1, \ldots, p_l$, such that all occurring numbers are between 0 and $-1 + \prod_{i=1}^{l} p_i$. During the computation we perform all basic operations modulo p_i for all $1 \le i \le l$. For the final result we can recover the real integers behind by applying the Chinese remainder theorem. For our example of the IMF we choose $l = 3$, $p_1 = 2^{63} - 25$, $p_2 = 2^{63} - 165$, and $p_3 = 2^{63} - 259$.[7]

The number of losing coalitions is given by $\sum_{x=0}^{q-1} c(x)$ and the number of winning coalitions is given by $\sum_{x=q}^{C} c(x)$. Since the total number of coalitions is 2^n, both numbers can be determined in $O(n\Delta)$ time and $O(\Delta + n)$ space.

For the computation of the Banzhaf index we need to know either the number $c^w(x)$ of coalitions with weight sum x that contain player i or the number $c^{wo}(x)$ of coalitions with weight sum x that do not contain player i. For a fixed player i we set $c^{wo}(x) = 0$ for $0 \le x < w_i$. By looping from w_i to $q-1$ we can recursively compute $c^{wo}(x) = c(x) - c^{wo}(x - w_i)$, so that $\mathrm{Bz}_i^a(v) = \frac{1}{2^{n-1}} \sum_{x=q-w_i}^{q-1} c^{wo}(x)$. Alternatively, we set $c^w(x) = c(x)$ for all $C - w_i < x \le C$ and recursively compute $c^w(x) = c(x) - c^w(x + w_i)$ by looping from $C - w_i$ to q, so that $\mathrm{Bz}_i^a(v) = \frac{1}{2^{n-1}} \sum_{x=q}^{q+w_i-1} c^w(x)$.

Theorem 1. *The number of winning, losing coalitions and the Banzhaf indices of all players of a weighted game v can be computed in $O(n\Delta)$ time and $O(\Delta + n)$ space.*

3.2. **Counting coalitions per weight and size.** By $c(x, s)$ we denote the number of coalitions of weight x and cardinality s (for a given weighted game v). Algorithm 1 and Algorithm 2 can be easily adopted to this end. The running time and the memory requirements both increase by a factor of n, since $0 \le s \le n$. We remark $c(x, s) = 0$ for $x > \sum_{j=1}^{s} w_j$ or $x < \sum_{j=n-s+1}^{n} w_j$, assuming $w_1 \ge \cdots \ge w_n$.[8] These known values can be taken into account in the boundaries of the for-loops to save time and memory. By extending the definition and recursion for $c^{wo}(x)$, $c^w(x)$ to $c^{wo}(x, s)$, $c^w(x, s)$, we can state

$$\mathrm{SSI}_i(v) = \sum_{s=0}^{n-1} s!(n-s-1)! \cdot \sum_{x=q-w_i}^{q-1} c^{wo}(x, s) \text{ and } \mathrm{SSI}_i(v) = \sum_{s=0}^{n-1} s!(n-s-1)! \cdot \sum_{x=q}^{q+w_i-1} c^{w}(x, s+1).$$

Of course we can precompute the factorials and the product of the $n-1$ pairs of factorials. In our fixed-precision arithmetic we first compute the sums over the c^{wo} or c^w and then switch to arbitrary-precision arithmetic.[9]

Theorem 2. *The* SSI *indices of all players of a weighted game v can be computed in $O(n^2\Delta)$ time and $O(n\Delta)$ space.*

3.3. **Intersections of weighted games.** Some real-world voting systems are expressed as the intersection of, say k, weighted voting games $v_1, \ldots, v_k$, i.e., a coalition is winning if and only if it is winning in all sub-games $v_1, \ldots, v_k$. Let $C_1, \ldots, C_k$ be the weights sums and $q_1, \ldots, q_k$ be the quotas of the sub-games. By easily extending our counting functions $c(x)$ and $c(x, s)$ to $c(x_1, \ldots, x_k)$ and $c(x_1, \ldots, x_k, s)$ we can go along the same lines as in the previous two subsections and obtain algorithms with the same complexity bounds if we formally set $\Delta = \min\left\{\prod_{i=1}^{k} q_i, \prod_{i=1}^{k} C_i - q_i + 1\right\}$. This number may grow very quickly even for moderate values of k, so that it may be crucial to choose a representation with a small number k of sub-games. We remark that the smallest possible integer k (for a simple game) is called *dimension*.

3.4. **Computational results.** We have applied the described algorithms for the four weighted voting games arising from the two different sets of voting weights of the IMF in 2015 and 2016, see tables 2-5, and quotas of either 85% or 50% of the respective weight sums.[10]All computations were performed on an Intel(R) Core(TM) i7-3720QM cpu with a clock speed of 2.60 GHz and 8 GB RAM. As a

[7]Choosing primes of the form $2^{63} - x$ for small x, has the advantage that the computations can be performed using the standard, simple data type `unsigned long` in `C++`. Our choices are indeed the largest possibilities, see e.g. `https://primes.utm.edu/lists/2small/0bit.html`. We remark that a naïve checking of the primality of the p_i was performed in 41 seconds. We implement $a = b + c \bmod p$ as $a = b + c$ and if $a \ge p$ then $a -= p$.

[8]The players can be sorted in $O(n + \Delta)$ time and space in a preprocessing step.

[9]We remark $s!(n-s-1)! < 2^{n \log_2 n}$ for $n > 1$.

[10]According to the *type* of the decision different values for q are used, see e.g. [7].

a general-purpose programming language we have chosen C++ and used the CLN-library[11] for the arbitrary-precision arithmetic parts.

For 2016 and super-majority, i.e., $q = 85\%$, Algorithm 1 needed 4.73 seconds and Algorithm 2 needed 0.67 seconds. The acceleration factor for using the described tailored fixed-precision arithmetic over an arbitrary-precision arithmetic is slightly larger than 6. Using pointers instead of the STL class `vector` results in a speed-up of roughly 2. The number of winning coalitions is given by 4506727722110247822679513808100007271801182981184082. The entire Banzhaf computation, based on Algorithm 2, for all players was performed in less than 3 seconds. The corresponding SSI computation took less than 7 minutes.

We have listed the power distributions for the years 2015 and 2016, the cases of super-majority and simple majority, the power indices Bz and SSI in tables 6-9 using a precision of five decimal digits for the output. The power of a few countries seem to coincide, which is a numerical artefact, except for France ($i = 58$) and the United Kingdom ($i = 179$). To be more precise, those two countries have the same weights in both 2015 and 2016, so that they are equivalent for all values of the quota q. For super-majority and simple majority all other countries are inequivalent, which may be seen at the exact values of either the Banzhaf or the Shapley-Shubik index. We have written out the exact integers Bz_i^a for the super-majority case in tables 10-13. As predicted by theory, all values have the same parity. For the exact values of $n! \cdot \mathrm{SSI}_i$ we have

7943491105336407342763365681397386123932473437094983074178260503548102866231696606
6280159129991609891845042632480167905404863728522082305141284091873293991567044167
8913841818287049871073575292808579836984419649634709080444217051613536750323984357
7219072966973233208480541488235675648000000000000000000000000000000000000000000

as an example for the United States in 2015 in the super-majority case. The remaining exact values can be obtained from the author upon request.

Having a closer look at the different power distributions we observe that the choice of the power index or the quota as well as the modified weights have a significant impact. The dominance of the United states has further increased from 2015 to 2016. Interestingly enough, the Banzhaf power in the super-majority cases shows almost no difference to, e.g., Japan, which is different for the Shapley-Shubik index. To obtain a more complete, but still compact, overview about the differences we have introduced $\Delta P_y^q(I) := \sum_{i\in I}\left|\mathrm{Bz}_i(v_y^q) - \mathrm{SSI}_i(v_y^q)\right|$, $\Delta\,\mathrm{Bz}^q(I) := \sum_{i\in I}\left|\mathrm{Bz}_i(v_{2015}^q) - \mathrm{Bz}_i(v_{2016}^q)\right|$, and $\Delta\,\mathrm{SSI}^q(I) := \sum_{i\in I}\left|\mathrm{SSI}_i(v_{2015}^q) - \mathrm{SSI}_i(v_{2016}^q)\right|$, where $q \in \{85\%, 50\%\}$, $y \in \{2015, 2016\}$, and v_y^q denotes the corresponding weighted game of the IMF. We evaluate those values on the entire set of countries N and on all countries except the biggest five (United States, Japan, Germany, France, United Kingdom) $\underline{N}$, see Table 1.

Table 1. Differences in power between years and power indices

set of countries I	$\Delta P_{2015}^{85\%}$	$\Delta P_{2016}^{85\%}$	$\Delta\,\mathrm{Bz}^{85\%}$	$\Delta\,\mathrm{SSI}^{85\%}$	$\Delta P_{2015}^{50\%}$	$\Delta P_{2016}^{50\%}$	$\Delta\,\mathrm{Bz}^{50\%}$	$\Delta\,\mathrm{SSI}^{50\%}$
N	54.33%	51.98%	19.90%	27.00%	9.46%	32.21%	46.52%	34.68%
$\underline{N}$	28.48%	26.17%	18.45%	17.67%	2.68%	9.70%	18.35%	18.00%

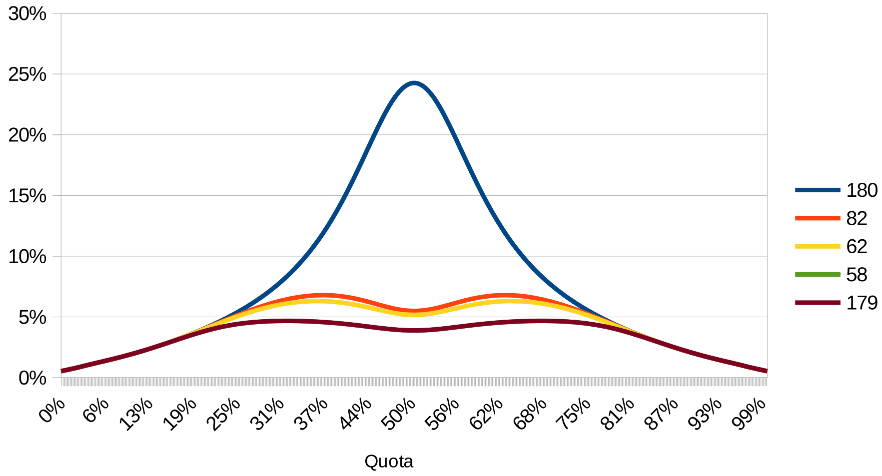
The stated running times can be easily extrapolated to other examples. If we assume a hypothetical IMF consisting of 1000 members whose weights are of similar magnitude as in the 2016 example, then both n and Δ increase by a factor of 1000/188. Instead of three primes we would need 16 primes, so that the computation of the Banzhaf indices in the super-majority case would took approximately 8 minutes while the SSI indices may be computed in 4 days.[12]

The minuscule running time for the Banzhaf index obviously allows more sophisticated applications where the power index computation is performed several times. Trying to heuristically solve the inverse power index problem, where weights need to be found whose power distribution is close to a given target distribution, is just an example. Here we have computed the Banzhaf power distribution of the IMF when the quota changes from 0% to 100% in steps of 0.1%, i.e., 1001 evaluations have been performed, as another possible application. In figures 1-2 we have depicted the corresponding power for the five most powerful countries (United States $i = 180$, Japan $i = 82$, Germany $i = 62$, France $i = 58$, United Kingdom $i = 179$). We can see that the respective Banzhaf indices are rather volatile

[11]CLN - Class Library for Numbers, available at `http://www.ginac.de/CLN`.

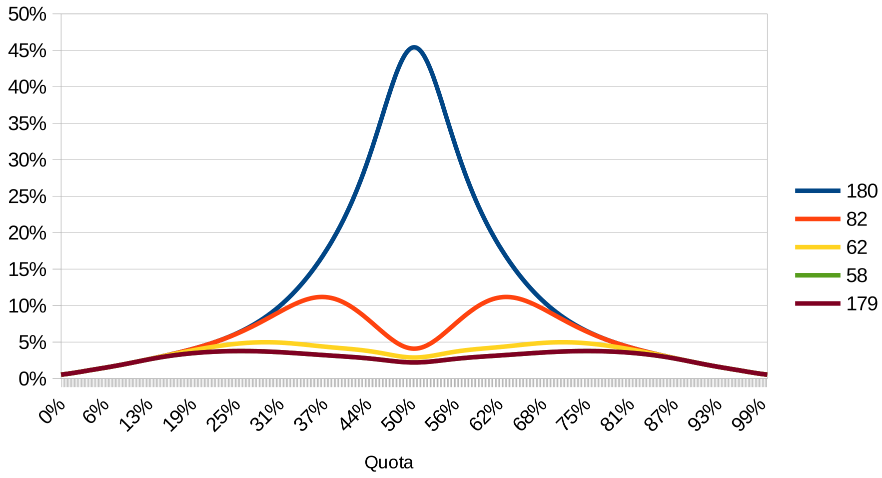
[12]In the latter case the memory requirements might become a serious issue if the computations for the different primes are not performed consecutively. Computing the intermediate results for each prime in parallel is the better option anyway. Using 16 computers (or cores) the mentioned times reduce to seconds and less than 7 hours, respectively.

Figure 1. Banzhaf power distribution of the IMF in 2015 for the five largest countries with variable quota



with respect to changes of the quota. The difference between the respective Banzhaf power shares is negligible for Japan and Germany, while there is no difference between France and the United Kingdom, for all values of the quota q. For an extreme quota of 0% or 100% all countries obtain exactly the same Banzhaf power share. For quotas below 15% or above 85% there is almost no difference in power for the five largest countries. However, there is a critical interval, say q between 25% and 75%, where the relative power distribution between the five largest countries is very sensitive to changes of the quota. The United States most intensive benefit from quotas around 50%. In 2016 also the Banzhaf power share of Japan is very sensitive to changes of the quota. Instead of 50% a quota of roughly 65% would be rather favorable for them.

Figure 2. Banzhaf power distribution of the IMF in 2016 for the five largest countries with variable quota



4. Conclusions

Nevertheless the computation of both the Banzhaf and the Shapley-Shubik index is NP-hard for weighted voting games, we have demonstrated that in practice it is not too hard to compute the exact values if the considered games are not *too large*. In the used sense, the current IMF voting system is definitely not too large since the Banzhaf indices can be computed in seconds and the Shapley-Shubik indices can be computed in a few minutes. For weighted games of that magnitude no approximations to the real values are necessary.

Even more, an efficient computation does not rely on sophisticated algorithms but low-level details in order to gain speed-up factors. For $C \ll 2^n$, which should be the case for all non-tiny real–world examples, the use of generating function approaches yields no benefit, although being a common topic in the literature. The used underlying idea of counting coalitions per weight and size by a simple recursion was just enhanced by allowing the reverse direction starting from the weight sum C for quotas larger than 50%. There is a single small insight that allows to recover those counts for the cases where a certain player is either assumed to be part or not to be part of the counted coalitions more efficiently than a direct enumeration. Using this approach the complexity for computing the considered indices are (up to a small constant) the same for a single player and all players.

For our real–world example of the IMF, the resulting power distributions are rather different from the weight shares and between diverse power indices like the Banzhaf and the Shapley-Shubik index. We suspect that this is not a numerical artefact of this specific example, so that it might be a good idea to compute several power indices to get a more comprehensive view whenever the considered committee has some non-negligible impact.

The distribution of the Banzhaf power shares is rather sensitive to changes of the quota and there are clear incentives for the few largest countries to alter them in their sense. The conclusion that may be drawn from that fact is debatable and the choice of the quota should indeed obtain more consideration.

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Tables of voting weights and the power distribution

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Table 2. Voting weights in the IMF – part 1

index	member/year	2015		2016	
1	Afghanistan, Islamic Republic of	2357	0.094%	2665	0.075%
2	Albania	1338	0.053%	2439	0.068%
3	Algeria	13285	0.527%	13593	0.380%
4	Angola	3601	0.143%	3909	0.109%
5	Antigua and Barbuda	873	0.035%	1246	0.035%
6	Argentina	21909	0.869%	22217	0.622%
7	Armenia	1658	0.066%	1966	0.055%
8	Australia	33102	1.313%	66770	1.869%
9	Austria	21877	0.868%	22185	0.621%
10	Azerbaijan	2347	0.093%	2655	0.074%
11	Bahamas, The	2041	0.081%	2870	0.080%
12	Bahrain	2088	0.083%	2396	0.067%
13	Bangladesh	6071	0.241%	6379	0.179%
14	Barbados	1413	0.056%	1991	0.056%
15	Belarus	4602	0.183%	4910	0.137%
16	Belgium	46790	1.856%	47098	1.318%
17	Belize	926	0.037%	1313	0.037%
18	Benin	1357	0.054%	1665	0.047%
19	Bhutan	801	0.032%	1109	0.031%
20	Bolivia	2453	0.097%	2761	0.077%
21	Bosnia and Herzegovina	2429	0.096%	2737	0.077%
22	Botswana	1616	0.064%	3018	0.084%
23	Brazil	43243	1.716%	43551	1.219%
24	Brunei Darussalam	2890	0.115%	3198	0.090%
25	Bulgaria	7140	0.283%	7448	0.208%
26	Burkina Faso	1340	0.053%	1648	0.046%
27	Burundi	1508	0.060%	1816	0.051%
28	Cabo Verde	850	0.034%	1158	0.032%
29	Cambodia	1613	0.064%	2796	0.078%
30	Cameroon	2595	0.103%	2903	0.081%
31	Canada	64430	2.556%	111285	3.115%
32	Central African Republic	1295	0.051%	1603	0.045%
33	Chad	1404	0.056%	2448	0.069%
34	Chile	9299	0.369%	18489	0.517%
35	China	95997	3.809%	96305	2.695%
36	Colombia	8478	0.336%	21491	0.601%
37	Comoros	827	0.033%	1224	0.034%
38	Congo, Democratic Republic of the	6068	0.241%	6376	0.178%
39	Congo, Republic of	1584	0.063%	1892	0.053%
40	Costa Rica	2379	0.094%	2687	0.075%
41	Côte d'Ivoire	3990	0.158%	4298	0.120%
42	Croatia	4389	0.174%	4697	0.131%
43	Cyprus	2320	0.092%	4084	0.114%
44	Czech Republic	10760	0.427%	11068	0.310%
45	Denmark	19652	0.780%	35440	0.992%
46	Djibouti	897	0.036%	1364	0.038%

Table 3. Voting weights in the IMF – part 2

index	member/year	2015		2016	
47	Dominica	820	0.033%	1161	0.032%
48	Dominican Republic	2927	0.116%	3235	0.091%
49	Ecuador	4216	0.167%	4524	0.127%
50	Egypt	10175	0.404%	10483	0.293%
51	El Salvador	2451	0.097%	2759	0.077%
52	Equatorial Guinea	1261	0.050%	1569	0.044%
53	Eritrea	897	0.036%	1205	0.034%
54	Estonia	1677	0.067%	3482	0.097%
55	Ethiopia	2075	0.082%	4053	0.113%
56	Fiji, Republic of	1441	0.057%	1749	0.049%
57	Finland	13376	0.531%	13684	0.383%
58	France	108123	4.290%	108431	3.035%
59	Gabon	2281	0.090%	3206	0.090%
60	Gambia, The	1049	0.042%	1357	0.038%
61	Georgia	2241	0.089%	3150	0.088%
62	Germany	146393	5.808%	146701	4.106%
63	Ghana	4428	0.176%	4736	0.133%
64	Greece	11756	0.466%	25335	0.709%
65	Grenada	855	0.034%	1163	0.033%
66	Guatemala	2840	0.113%	3148	0.088%
67	Guinea	1809	0.072%	3188	0.089%
68	Guinea-Bissau	880	0.035%	1188	0.033%
69	Guyana	1647	0.065%	2864	0.080%
70	Haiti	1557	0.062%	1865	0.052%
71	Honduras	2033	0.081%	2341	0.066%
72	Hungary	11122	0.441%	20446	0.572%
73	Iceland	1914	0.076%	4264	0.119%
74	India	58953	2.339%	59261	1.659%
75	Indonesia	21531	0.854%	21839	0.611%
76	Iran, Islamic Republic of	15710	0.623%	16018	0.448%
77	Iraq	12622	0.501%	12930	0.362%
78	Ireland	13314	0.528%	13622	0.381%
79	Israel	11349	0.450%	20255	0.567%
80	Italy	79561	3.156%	79869	2.235%
81	Jamaica	3473	0.138%	4875	0.136%
82	Japan	157023	6.230%	309251	8.655%
83	Jordan	2443	0.097%	4477	0.125%
84	Kazakhstan	5016	0.199%	5324	0.149%
85	Kenya	3452	0.137%	3760	0.105%
86	Kiribati	794	0.032%	1102	0.031%
87	Korea	34402	1.365%	34710	0.971%
88	Kosovo	1328	0.053%	1636	0.046%
89	Kuwait	14549	0.577%	14857	0.416%
90	Kyrgyz Republic	1626	0.065%	1934	0.054%
91	Lao People's Democratic Republic	1267	0.050%	1575	0.044%
92	Latvia	2159	0.086%	2467	0.069%
93	Lebanon	3402	0.135%	3710	0.104%
94	Lesotho	1087	0.043%	1395	0.039%

Table 4. Voting weights in the IMF – part 3

index	member/year	2015		2016	
95	Liberia	2030	0.081%	2338	0.065%
96	Libya	11975	0.475%	12283	0.344%
97	Lithuania	2577	0.102%	5462	0.153%
98	Luxembourg	4925	0.195%	5233	0.146%
99	Macedonia, former Yugoslav Republic of	1427	0.057%	2449	0.069%
100	Madagascar	1960	0.078%	2268	0.063%
101	Malawi	1432	0.057%	2434	0.068%
102	Malaysia	18477	0.733%	18785	0.526%
103	Maldives	838	0.033%	1146	0.032%
104	Mali	1671	0.066%	1979	0.055%
105	Malta	1758	0.070%	2066	0.058%
106	Marshall Islands	773	0.031%	1081	0.030%
107	Mauritania	1382	0.055%	1690	0.047%
108	Mauritius	1754	0.070%	2468	0.069%
109	Mexico	36995	1.468%	90173	2.524%
110	Micronesia, Federated States of	789	0.031%	1097	0.031%
111	Moldova	1970	0.078%	2278	0.064%
112	Mongolia	1249	0.050%	1557	0.044%
113	Montenegro	1013	0.040%	1651	0.046%
114	Morocco	6620	0.263%	9990	0.280%
115	Mozambique	1874	0.074%	2182	0.061%
116	Myanmar	3322	0.132%	6214	0.174%
117	Namibia	2103	0.083%	2411	0.067%
118	Nepal	1451	0.058%	1759	0.049%
119	Netherlands	52362	2.077%	88411	2.474%
120	New Zealand	9684	0.384%	9992	0.280%
121	Nicaragua	2038	0.081%	2346	0.066%
122	Niger	1396	0.055%	1704	0.048%
123	Nigeria	18270	0.725%	18578	0.520%
124	Norway	19575	0.777%	19883	0.556%
125	Oman	3108	0.123%	3416	0.096%
126	Pakistan	11075	0.439%	21356	0.598%
127	Palau	769	0.031%	1077	0.030%
128	Panama	2804	0.111%	3112	0.087%
129	Papua New Guinea	2054	0.081%	2362	0.066%
130	Paraguay	1737	0.069%	2045	0.057%
131	Peru	7122	0.283%	14391	0.403%
132	Philippines	10931	0.434%	11239	0.315%
133	Poland	17622	0.699%	42000	1.176%
134	Portugal	11035	0.438%	21647	0.606%
135	Qatar	3764	0.149%	4072	0.114%
136	Romania	11040	0.438%	19160	0.536%
137	Russian Federation	60192	2.388%	60500	1.693%
138	Rwanda	1539	0.061%	1847	0.052%
139	Samoa	854	0.034%	1162	0.033%
140	San Marino	962	0.038%	1538	0.043%
141	São Tomé and Príncipe	812	0.032%	1194	0.033%

Table 5. Voting weights in the IMF – part 4

index	member/year	2015		2016	
142	Saudi Arabia	70593	2.801%	70901	1.984%
143	Senegal	2356	0.093%	2664	0.075%
144	Serbia	5415	0.215%	7594	0.213%
145	Seychelles	847	0.034%	1275	0.036%
146	Sierra Leone	1775	0.070%	2083	0.058%
147	Singapore	14818	0.588%	15126	0.423%
148	Slovak Republic	5013	0.199%	5321	0.149%
149	Slovenia	3488	0.138%	3796	0.106%
150	Solomon Islands	842	0.033%	1150	0.032%
151	Somalia	1180	0.047%	1488	0.042%
152	South Africa	19423	0.771%	19731	0.552%
153	South Sudan, Republic of	1968	0.078%	2276	0.064%
154	Spain	40972	1.626%	96401	2.698%
155	Sri Lanka	4872	0.193%	5180	0.145%
156	St. Kitts and Nevis	827	0.033%	1135	0.032%
157	St. Lucia	891	0.035%	1199	0.034%
158	St. Vincent and the Grenadines	821	0.033%	1129	0.032%
159	Sudan	2435	0.097%	2743	0.077%
160	Suriname	1659	0.066%	1967	0.055%
161	Swaziland	1245	0.049%	1831	0.051%
162	Sweden	24693	0.980%	45346	1.269%
163	Switzerland	35323	1.401%	58757	1.645%
164	Syrian Arab Republic	3674	0.146%	3982	0.111%
165	Tajikistan	1608	0.064%	1916	0.054%
166	Tanzania	2727	0.108%	3035	0.085%
167	Thailand	15143	0.601%	15451	0.432%
168	Timor-Leste	846	0.034%	1154	0.032%
169	Togo	1472	0.058%	1780	0.050%
170	Tonga	807	0.032%	1115	0.031%
171	Trinidad and Tobago	4094	0.162%	4402	0.123%
172	Tunisia	3603	0.143%	3911	0.109%
173	Turkey	15296	0.607%	15604	0.437%
174	Turkmenistan	1490	0.059%	3432	0.096%
175	Tuvalu	756	0.030%	1064	0.030%
176	Uganda	2543	0.101%	2851	0.080%
177	Ukraine	14458	0.574%	21164	0.592%
178	United Arab Emirates	8263	0.328%	8571	0.240%
179	United Kingdom	108123	4.290%	108431	3.035%
180	United States	421962	16.741%	830988	23.258%
181	Uruguay	3803	0.151%	5337	0.149%
182	Uzbekistan	3494	0.139%	3802	0.106%
183	Vanuatu	908	0.036%	1216	0.034%
184	Venezuela, República Bolivariana de	27329	1.084%	27637	0.774%
185	Vietnam	5345	0.212%	5653	0.158%
186	Yemen, Republic of	3173	0.126%	3481	0.097%
187	Zambia	5629	0.223%	5937	0.166%
188	Zimbabwe	4272	0.169%	4580	0.128%
	total	2520571	100.000%	3572928	100.000%

Table 6. Voting power in the IMF – part 1

	supermajority				simple majority			
	Bz_i		SSI_i		Bz_i		SSI_i	
i	2015	2016	2015	2016	2015	2016	2015	2016
1	0.153%	0.126%	0.088%	0.074%	0.086%	0.056%	0.088%	0.067%
2	0.087%	0.115%	0.050%	0.068%	0.049%	0.051%	0.050%	0.062%
3	0.845%	0.634%	0.497%	0.378%	0.483%	0.286%	0.501%	0.345%
4	0.234%	0.184%	0.134%	0.108%	0.131%	0.082%	0.135%	0.099%
5	0.057%	0.059%	0.032%	0.034%	0.032%	0.026%	0.033%	0.032%
6	1.342%	1.021%	0.824%	0.621%	0.796%	0.467%	0.828%	0.565%
7	0.108%	0.093%	0.062%	0.054%	0.060%	0.041%	0.062%	0.050%
8	1.882%	2.522%	1.254%	1.906%	1.202%	1.390%	1.256%	1.714%
9	1.340%	1.019%	0.823%	0.620%	0.795%	0.466%	0.827%	0.564%
10	0.153%	0.125%	0.087%	0.074%	0.085%	0.056%	0.088%	0.067%
11	0.133%	0.135%	0.076%	0.079%	0.074%	0.060%	0.077%	0.073%
12	0.136%	0.113%	0.078%	0.066%	0.076%	0.050%	0.078%	0.061%
13	0.393%	0.300%	0.226%	0.177%	0.221%	0.134%	0.228%	0.162%
14	0.092%	0.094%	0.052%	0.055%	0.051%	0.042%	0.053%	0.050%
15	0.298%	0.231%	0.171%	0.136%	0.167%	0.103%	0.173%	0.124%
16	2.349%	1.982%	1.789%	1.331%	1.698%	0.986%	1.785%	1.204%
17	0.060%	0.062%	0.034%	0.036%	0.034%	0.028%	0.035%	0.033%
18	0.088%	0.078%	0.050%	0.046%	0.049%	0.035%	0.051%	0.042%
19	0.052%	0.052%	0.030%	0.031%	0.029%	0.023%	0.030%	0.028%
20	0.159%	0.130%	0.091%	0.076%	0.089%	0.058%	0.092%	0.070%
21	0.158%	0.129%	0.090%	0.076%	0.088%	0.058%	0.091%	0.069%
22	0.105%	0.142%	0.060%	0.084%	0.059%	0.064%	0.061%	0.076%
23	2.249%	1.863%	1.649%	1.229%	1.570%	0.912%	1.647%	1.112%
24	0.188%	0.151%	0.107%	0.089%	0.105%	0.067%	0.109%	0.081%
25	0.461%	0.350%	0.266%	0.207%	0.259%	0.157%	0.268%	0.189%
26	0.087%	0.078%	0.050%	0.046%	0.049%	0.035%	0.050%	0.042%
27	0.098%	0.086%	0.056%	0.050%	0.055%	0.038%	0.057%	0.046%
28	0.055%	0.055%	0.032%	0.032%	0.031%	0.024%	0.032%	0.029%
29	0.105%	0.132%	0.060%	0.077%	0.059%	0.059%	0.061%	0.071%
30	0.169%	0.137%	0.096%	0.080%	0.094%	0.061%	0.097%	0.073%
31	2.667%	3.073%	2.492%	3.249%	2.335%	2.266%	2.474%	2.885%
32	0.084%	0.076%	0.048%	0.044%	0.047%	0.034%	0.049%	0.041%
33	0.091%	0.115%	0.052%	0.068%	0.051%	0.052%	0.053%	0.062%
34	0.598%	0.856%	0.347%	0.516%	0.338%	0.389%	0.350%	0.470%
35	2.828%	2.967%	3.796%	2.790%	3.462%	1.979%	3.730%	2.488%
36	0.546%	0.989%	0.316%	0.600%	0.308%	0.452%	0.319%	0.546%
37	0.054%	0.058%	0.031%	0.034%	0.030%	0.026%	0.031%	0.031%
38	0.393%	0.300%	0.226%	0.177%	0.221%	0.134%	0.228%	0.161%
39	0.103%	0.089%	0.059%	0.052%	0.058%	0.040%	0.059%	0.048%
40	0.155%	0.127%	0.088%	0.074%	0.086%	0.057%	0.089%	0.068%
41	0.259%	0.202%	0.148%	0.119%	0.145%	0.090%	0.150%	0.109%
42	0.285%	0.221%	0.163%	0.130%	0.159%	0.099%	0.165%	0.119%
43	0.151%	0.192%	0.086%	0.113%	0.084%	0.086%	0.087%	0.103%
44	0.690%	0.518%	0.402%	0.308%	0.391%	0.233%	0.405%	0.281%
45	1.218%	1.566%	0.738%	0.996%	0.714%	0.744%	0.742%	0.903%
46	0.058%	0.064%	0.033%	0.038%	0.033%	0.029%	0.034%	0.035%
47	0.053%	0.055%	0.030%	0.032%	0.030%	0.024%	0.031%	0.029%

Table 7. Voting power in the IMF – part 2

	supermajority				simple majority			
	Bz_i		SSI_i		Bz_i		SSI_i	
i	2015	2016	2015	2016	2015	2016	2015	2016
48	0.190%	0.152%	0.109%	0.090%	0.106%	0.068%	0.110%	0.082%
49	0.274%	0.213%	0.157%	0.125%	0.153%	0.095%	0.158%	0.115%
50	0.653%	0.491%	0.380%	0.291%	0.370%	0.221%	0.383%	0.266%
51	0.159%	0.130%	0.091%	0.076%	0.089%	0.058%	0.092%	0.070%
52	0.082%	0.074%	0.047%	0.043%	0.046%	0.033%	0.047%	0.040%
53	0.058%	0.057%	0.033%	0.033%	0.033%	0.025%	0.034%	0.030%
54	0.109%	0.164%	0.062%	0.096%	0.061%	0.073%	0.063%	0.088%
55	0.135%	0.191%	0.077%	0.112%	0.075%	0.085%	0.078%	0.103%
56	0.094%	0.082%	0.054%	0.048%	0.052%	0.037%	0.054%	0.044%
57	0.851%	0.639%	0.500%	0.381%	0.486%	0.288%	0.504%	0.347%
58	2.842%	3.057%	4.314%	3.161%	3.888%	2.212%	4.221%	2.809%
59	0.148%	0.151%	0.085%	0.089%	0.083%	0.067%	0.086%	0.081%
60	0.068%	0.064%	0.039%	0.038%	0.038%	0.029%	0.039%	0.034%
61	0.146%	0.148%	0.083%	0.087%	0.081%	0.066%	0.084%	0.080%
62	2.850%	3.173%	6.017%	4.364%	5.172%	2.890%	5.803%	3.834%
63	0.287%	0.223%	0.165%	0.131%	0.161%	0.100%	0.166%	0.120%
64	0.751%	1.155%	0.439%	0.709%	0.427%	0.532%	0.443%	0.644%
65	0.056%	0.055%	0.032%	0.032%	0.031%	0.024%	0.032%	0.029%
66	0.184%	0.148%	0.106%	0.087%	0.103%	0.066%	0.107%	0.080%
67	0.118%	0.150%	0.067%	0.088%	0.066%	0.067%	0.068%	0.081%
68	0.057%	0.056%	0.033%	0.033%	0.032%	0.025%	0.033%	0.030%
69	0.107%	0.135%	0.061%	0.079%	0.060%	0.060%	0.062%	0.072%
70	0.101%	0.088%	0.058%	0.052%	0.057%	0.039%	0.058%	0.047%
71	0.132%	0.110%	0.076%	0.065%	0.074%	0.049%	0.076%	0.059%
72	0.712%	0.943%	0.416%	0.571%	0.404%	0.430%	0.419%	0.519%
73	0.124%	0.201%	0.071%	0.118%	0.070%	0.090%	0.072%	0.108%
74	2.596%	2.341%	2.272%	1.685%	2.137%	1.237%	2.259%	1.519%
75	1.322%	1.004%	0.810%	0.610%	0.782%	0.459%	0.814%	0.555%
76	0.991%	0.745%	0.589%	0.446%	0.571%	0.337%	0.593%	0.407%
77	0.805%	0.604%	0.472%	0.360%	0.459%	0.272%	0.476%	0.328%
78	0.847%	0.636%	0.498%	0.379%	0.484%	0.287%	0.502%	0.346%
79	0.726%	0.934%	0.424%	0.565%	0.412%	0.426%	0.427%	0.515%
80	2.781%	2.768%	3.109%	2.295%	2.877%	1.655%	3.072%	2.056%
81	0.225%	0.229%	0.129%	0.135%	0.126%	0.103%	0.130%	0.123%
82	2.851%	3.198%	6.511%	10.262%	5.494%	4.103%	6.251%	8.370%
83	0.159%	0.211%	0.091%	0.124%	0.089%	0.094%	0.092%	0.113%
84	0.325%	0.251%	0.187%	0.148%	0.182%	0.112%	0.188%	0.135%
85	0.224%	0.177%	0.128%	0.104%	0.125%	0.079%	0.130%	0.095%
86	0.052%	0.052%	0.029%	0.030%	0.029%	0.023%	0.030%	0.028%
87	1.936%	1.537%	1.304%	0.975%	1.249%	0.728%	1.306%	0.885%
88	0.086%	0.077%	0.049%	0.045%	0.048%	0.034%	0.050%	0.041%
89	0.921%	0.692%	0.545%	0.414%	0.529%	0.312%	0.549%	0.377%
90	0.106%	0.091%	0.060%	0.054%	0.059%	0.041%	0.061%	0.049%
91	0.082%	0.074%	0.047%	0.044%	0.046%	0.033%	0.048%	0.040%
92	0.140%	0.116%	0.080%	0.068%	0.078%	0.052%	0.081%	0.062%
93	0.221%	0.175%	0.127%	0.103%	0.124%	0.078%	0.128%	0.094%
94	0.071%	0.066%	0.040%	0.039%	0.040%	0.029%	0.041%	0.035%

TABLE 8. Voting power in the IMF – part 3

	supermajority				simple majority			
	Bz_i		SSI_i		Bz_i		SSI_i	
i	2015	2016	2015	2016	2015	2016	2015	2016
95	0.132%	0.110%	0.075%	0.065%	0.074%	0.049%	0.076%	0.059%
96	0.765%	0.574%	0.448%	0.342%	0.435%	0.258%	0.451%	0.312%
97	0.167%	0.257%	0.096%	0.151%	0.094%	0.115%	0.097%	0.138%
98	0.319%	0.246%	0.183%	0.145%	0.179%	0.110%	0.185%	0.133%
99	0.093%	0.115%	0.053%	0.068%	0.052%	0.052%	0.054%	0.062%
100	0.127%	0.107%	0.073%	0.063%	0.071%	0.048%	0.074%	0.057%
101	0.093%	0.115%	0.053%	0.067%	0.052%	0.051%	0.054%	0.062%
102	1.151%	0.869%	0.694%	0.524%	0.671%	0.395%	0.698%	0.477%
103	0.054%	0.054%	0.031%	0.032%	0.030%	0.024%	0.031%	0.029%
104	0.109%	0.093%	0.062%	0.055%	0.061%	0.042%	0.063%	0.050%
105	0.114%	0.097%	0.065%	0.057%	0.064%	0.043%	0.066%	0.052%
106	0.050%	0.051%	0.029%	0.030%	0.028%	0.023%	0.029%	0.027%
107	0.090%	0.080%	0.051%	0.047%	0.050%	0.036%	0.052%	0.043%
108	0.114%	0.116%	0.065%	0.068%	0.064%	0.052%	0.066%	0.062%
109	2.037%	2.905%	1.405%	2.604%	1.343%	1.859%	1.406%	2.327%
110	0.051%	0.052%	0.029%	0.030%	0.029%	0.023%	0.030%	0.028%
111	0.128%	0.107%	0.073%	0.063%	0.072%	0.048%	0.074%	0.058%
112	0.081%	0.073%	0.046%	0.043%	0.045%	0.033%	0.047%	0.039%
113	0.066%	0.078%	0.038%	0.046%	0.037%	0.035%	0.038%	0.042%
114	0.428%	0.468%	0.247%	0.278%	0.241%	0.210%	0.249%	0.253%
115	0.122%	0.103%	0.070%	0.060%	0.068%	0.046%	0.070%	0.055%
116	0.216%	0.292%	0.124%	0.172%	0.121%	0.131%	0.125%	0.157%
117	0.137%	0.114%	0.078%	0.067%	0.076%	0.051%	0.079%	0.061%
118	0.094%	0.083%	0.054%	0.049%	0.053%	0.037%	0.054%	0.045%
119	2.479%	2.884%	2.009%	2.551%	1.899%	1.824%	2.001%	2.280%
120	0.622%	0.468%	0.362%	0.278%	0.352%	0.210%	0.364%	0.253%
121	0.132%	0.111%	0.076%	0.065%	0.074%	0.049%	0.076%	0.059%
122	0.091%	0.080%	0.052%	0.047%	0.051%	0.036%	0.052%	0.043%
123	1.140%	0.860%	0.686%	0.518%	0.664%	0.391%	0.690%	0.472%
124	1.214%	0.918%	0.735%	0.555%	0.711%	0.418%	0.739%	0.505%
125	0.202%	0.161%	0.116%	0.095%	0.113%	0.072%	0.117%	0.086%
126	0.709%	0.983%	0.414%	0.596%	0.402%	0.449%	0.417%	0.543%
127	0.050%	0.051%	0.029%	0.030%	0.028%	0.023%	0.029%	0.027%
128	0.182%	0.147%	0.104%	0.086%	0.102%	0.065%	0.105%	0.079%
129	0.133%	0.111%	0.076%	0.065%	0.075%	0.050%	0.077%	0.060%
130	0.113%	0.096%	0.065%	0.057%	0.063%	0.043%	0.065%	0.052%
131	0.460%	0.671%	0.265%	0.401%	0.259%	0.303%	0.268%	0.365%
132	0.700%	0.526%	0.408%	0.312%	0.397%	0.236%	0.412%	0.285%
133	1.102%	1.808%	0.661%	1.184%	0.640%	0.880%	0.665%	1.072%
134	0.707%	0.996%	0.412%	0.605%	0.401%	0.455%	0.416%	0.550%
135	0.244%	0.192%	0.140%	0.113%	0.137%	0.086%	0.141%	0.103%
136	0.707%	0.886%	0.412%	0.534%	0.401%	0.403%	0.416%	0.487%
137	2.613%	2.373%	2.321%	1.722%	2.182%	1.262%	2.307%	1.551%
138	0.100%	0.087%	0.057%	0.051%	0.056%	0.039%	0.058%	0.047%
139	0.056%	0.055%	0.032%	0.032%	0.031%	0.024%	0.032%	0.029%
140	0.063%	0.072%	0.036%	0.043%	0.035%	0.032%	0.036%	0.039%
141	0.053%	0.056%	0.030%	0.033%	0.030%	0.025%	0.030%	0.030%

Table 9. Voting power in the IMF – part 4

	supermajority				simple majority			
	Bz_i		SSI_i		Bz_i		SSI_i	
i	2015	2016	2015	2016	2015	2016	2015	2016
142	2.725%	2.609%	2.742%	2.028%	2.556%	1.474%	2.717%	1.821%
143	0.153%	0.126%	0.088%	0.074%	0.086%	0.056%	0.088%	0.067%
144	0.351%	0.357%	0.202%	0.211%	0.197%	0.160%	0.204%	0.192%
145	0.055%	0.060%	0.031%	0.035%	0.031%	0.027%	0.032%	0.032%
146	0.115%	0.098%	0.066%	0.058%	0.065%	0.044%	0.067%	0.053%
147	0.937%	0.704%	0.555%	0.421%	0.538%	0.318%	0.559%	0.384%
148	0.325%	0.250%	0.187%	0.147%	0.182%	0.112%	0.188%	0.135%
149	0.226%	0.179%	0.130%	0.105%	0.127%	0.080%	0.131%	0.096%
150	0.055%	0.054%	0.031%	0.032%	0.031%	0.024%	0.032%	0.029%
151	0.077%	0.070%	0.044%	0.041%	0.043%	0.031%	0.044%	0.038%
152	1.205%	0.911%	0.729%	0.551%	0.706%	0.415%	0.734%	0.501%
153	0.128%	0.107%	0.073%	0.063%	0.072%	0.048%	0.074%	0.058%
154	2.177%	2.968%	1.560%	2.793%	1.487%	1.981%	1.560%	2.491%
155	0.316%	0.244%	0.181%	0.144%	0.177%	0.109%	0.183%	0.131%
156	0.054%	0.053%	0.031%	0.031%	0.030%	0.024%	0.031%	0.029%
157	0.058%	0.057%	0.033%	0.033%	0.032%	0.025%	0.033%	0.030%
158	0.053%	0.053%	0.030%	0.031%	0.030%	0.024%	0.031%	0.029%
159	0.158%	0.129%	0.090%	0.076%	0.088%	0.058%	0.091%	0.069%
160	0.108%	0.093%	0.062%	0.054%	0.060%	0.041%	0.062%	0.050%
161	0.081%	0.086%	0.046%	0.051%	0.045%	0.039%	0.047%	0.046%
162	1.489%	1.924%	0.930%	1.281%	0.897%	0.950%	0.934%	1.158%
163	1.973%	2.328%	1.340%	1.671%	1.283%	1.226%	1.342%	1.505%
164	0.238%	0.188%	0.137%	0.110%	0.134%	0.084%	0.138%	0.101%
165	0.105%	0.090%	0.060%	0.053%	0.058%	0.040%	0.060%	0.048%
166	0.177%	0.143%	0.101%	0.084%	0.099%	0.064%	0.102%	0.077%
167	0.957%	0.719%	0.567%	0.430%	0.550%	0.325%	0.571%	0.392%
168	0.055%	0.054%	0.031%	0.032%	0.031%	0.024%	0.032%	0.029%
169	0.096%	0.084%	0.055%	0.049%	0.053%	0.037%	0.055%	0.045%
170	0.052%	0.053%	0.030%	0.031%	0.029%	0.023%	0.030%	0.028%
171	0.266%	0.207%	0.152%	0.122%	0.149%	0.093%	0.154%	0.111%
172	0.234%	0.184%	0.134%	0.108%	0.131%	0.082%	0.135%	0.099%
173	0.966%	0.726%	0.573%	0.435%	0.556%	0.328%	0.577%	0.396%
174	0.097%	0.162%	0.055%	0.095%	0.054%	0.072%	0.056%	0.087%
175	0.049%	0.050%	0.028%	0.029%	0.027%	0.022%	0.028%	0.027%
176	0.165%	0.134%	0.095%	0.079%	0.092%	0.060%	0.095%	0.072%
177	0.916%	0.975%	0.541%	0.591%	0.525%	0.445%	0.545%	0.538%
178	0.533%	0.402%	0.308%	0.238%	0.300%	0.180%	0.311%	0.217%
179	2.842%	3.057%	4.314%	3.161%	3.888%	2.212%	4.221%	2.809%
180	2.851%	3.198%	18.931%	20.552%	24.264%	45.408%	19.532%	29.305%
181	0.247%	0.251%	0.141%	0.148%	0.138%	0.112%	0.143%	0.135%
182	0.227%	0.179%	0.130%	0.105%	0.127%	0.080%	0.131%	0.096%
183	0.059%	0.057%	0.034%	0.034%	0.033%	0.026%	0.034%	0.031%
184	1.621%	1.252%	1.032%	0.774%	0.993%	0.581%	1.035%	0.703%
185	0.346%	0.266%	0.199%	0.157%	0.194%	0.119%	0.201%	0.143%
186	0.206%	0.164%	0.118%	0.096%	0.115%	0.073%	0.119%	0.088%
187	0.365%	0.279%	0.210%	0.165%	0.205%	0.125%	0.212%	0.150%
188	0.277%	0.216%	0.159%	0.127%	0.155%	0.096%	0.160%	0.116%

Table 10. Exact Banzhaf counts $\mathrm{Bz}_{\mathbf{i}}^{\mathbf{a}}$ for supermajority – part 1

i	2015	2016
1	49492696550469334288388474302188300702732503278942	176903926397724458899541247938651532746047517818906
2	28108818684726200071728584276680195673408532632506	161911654209258251928883011914306328209189949330986
3	273067686459011964864016012688469258989855194791048	894022914523707629419913395215803955826725915570768
4	75544231177486375978776859223371297251582054333762	259370900172430505766454804559359034798947935884810
5	18342422447942989488068768527179198893754345719314	82733927205579511449928590811181587395707525770476
6	433741858747716854195909526475817535173470581906728	1438072991668218360540956801914813269126044117351308
7	34827220802205252181870565955690262877158934233490	130525954991744089345193787551781828973012811299238
8	608355615407208421844899611782423706037812464078784	3554109746517798404853024001719094557801394784806464
9	433182870044323129114951461399711176952959411081012	1436106956965251152709612442958745097124836736161260
10	49283005345236751798983276620448523479102451082936	176240609357525813716577515545475648532489655198040
11	42864738912205592177496389792722579268856699369832	190500680136856761637048733570688650885901540528942
12	43850758455885814747859350839893223594872905963452	159058842854086308434367420892597348030339225405130
13	126981103851113231438250859484005597145786917207210	422702171034352800882553740542749645796961981262606
14	29683657447695375864399363718939502316029045938454	132185065852265487980401422802136319182059897021716
15	96444635233833481188612688283680819764478305210228	325639998090401901053487689403331852986834824794252
16	759097678746734836884323315755480826869874874425326	2793167523276080546458364914050593030600984647461638
17	19455761959935719977866329667841045679127995472904	87181924423240271839909708356772795604682325346728
18	28507789345030375152708673330225662529371453353754	110548355809084644822385758652463422320526696153662
19	16829899601638766359861993347371410650666478818128	73638414745411021335583392933993390894220786591860
20	51505540585623230167758938592922162559819362717262	183271486224930993363998639083642471351507889517350
21	51002362473137740132187832033553763107249588446574	181679644941715378420286313434583564485598714014822
22	33945572271601426703805861415125949584851120695898	200315331313091427450458314303074357862625102805912
23	726754733887620717507209621430284186494250141324258	2624647238299486195655732164291618128080904104245420
24	60663458855535188524981041217095772129837525292592	212250218546044259406241364028774311807493043718420
25	149076620619821626154374069054899834514667619210906	493160109266952002377850870425727145561149449532592
26	28150815957514997522594582448926474661249278580222	109419954342548217986460398689320358140356076426688
27	31678272140988691301178071813034847159103665129156	120570766697683856880979119192418096930670312307160
28	17859262918629192786168584675895889084393526298830	76891606544263926116936869857821286603334278092518
29	33882595640057256558166572877604802107208189423426	185592862331687555958418832460229675381502614337990
30	54482214301775562263962022540219773711109002545444	192689197192805681505312826962691667484571161465936
31	861801107408963078242493745604171364645766003005656	4329821441981762745938492120585762580549398688934530
32	27205857080040453614461131405757236729714350892358	106432961931312720728736156939232912352857776794814
33	29494683293591781782803413279082643271020444121920	162508742497033788697478131645323609539641382298902
34	193294951863880247537932957234235924307488089563436	1206184262774063458157220881438417065868208300759666
35	914069763507643309388664770743422000893797400468924	4180423179544972629576844266694840337387333726424916
36	176550844252180855245260966056871303574741707001364	1393360070480048036358668834241461862932187263216148
37	17376096284502694686549163684710986749876514974292	81273365229822355975892521495232249817499206435004
38	126918934379607418456760182715983019232203030831898	422504217377002706156077246588820813100825717545440
39	33273809778788364291922969434378859546262437743106	125614838925774030639349991615968388814798225098058
40	49954004087711010435778427309597267844134573800388	178363204434532533721206307106244714743477463154746
41	83673977492219995514323051389306912008266465360762	285134587505967951106373187599608865867576823141298
42	92002836574838726809193447169489151762397303979328	311546598088144771469437740555165260498535216556998
43	48716820636063120650288729950515927651347698822042	270962834596756689494207763451074016041725818937568
44	222847289610042165828194890756886793432262301679010	730307189069648489551446143352077294476026159230172
45	393576617133025460945604654629693026878034919636510	2206092526531846212355237223104989228779907316128206
46	18846581130575530182651709645454563684490631687646	90567631833337469924450997316356975356985309556256
47	17229044188849734251438725217475824444372553225338	77090779705804545592803924357452258038340519850530

TABLE 11. Exact Banzhaf counts $Bz_{\mathbf{i}}^{\mathbf{a}}$ for supermajority – part 2

i	2015	2016
48	61438462001559665113999981127215796585984210855648	214703239153916461314845491769597161273978081883676
49	88392867735151915207586461901703825857712334268926	300096595204460456827471144043187771118052667702764
50	211053904455260606918250786934031134685520223154414	692159614227603811889517072645010455131000993604518
51	51463609924728845688958697853656613736623382622284	183138834038284691902924536354058860926546225596384
52	26491860599355877436888623101254938092665600267626	104176078522686009873370603749878533351537159776114
53	18846581130575530182651709645454563684490631687646	80011960911102504338320129319511747538362140926976
54	35226046701012768753027566935295199597069035091906	231076410124979151881162099161280586981496993085196
55	43578036860352277311831574082303997667671986903024	268909591584396612960895259759692172143255227831394
56	30271565425364041223295271026912740220722945319812	116123838318055737270287596107957285794330767132264
57	274855137020666279269199374970231997864870048210246	899892218649456436875169655535237214322405426861162
58	918367775817962485090852104088421114577954611639702	4307308159804972843954035938800320763350232332137884
59	47898951240527749710017961061976900403572200768652	212780609094147596489991666417361711985637566370182
60	22039387529986534774087832506935106979681085495194	90102931184707889845392995440248071448484616882016
61	47060053966802681235618297814431692958818523244730	209067786668135323478445462486058449517906155000724
62	921127579853676641938863736581916284809724194641904	4471335071569148453877453107882772577536377224847344
63	92816362204161041934385362686097304932839374037648	314127415054053676866192769134807041047825429859600
64	242795054330860281325844430439489794822021757424122	1627415384966156569681074205771907540286261524762098
65	17964298209927657212291046379600863523460447370016	77223561693265480857675688245588580490558673933052
66	59616058340924379189331312394702299379953429226016	208935182064119182684976187818046396539747608364040
67	37996567309495184690084153193523420533122769559124	211587224405055108537163213838518473363899992728608
68	18489469560826214464573042797451899659361058052028	78883328350911004500335640950452759348729730821392
69	34596317270458725718320202780321217720145985946766	190102760971609859913108584067373791113321434932918
70	32706990229088039423540791358014498797704798696054	123822891350483667731643346703205954525160393149032
71	42696898665119331260060769898879029510028634632538	155409764606227699866930321669743573947464848841170
72	230117024395615612691963878945081408176408504340170	1328611389034414753225126922717140527546539672816046
73	40200036881623771794709307389561142981024679214696	282883267342459328161084725803150306408378366958534
74	838779547213705965890997275773362318112676568719436	3298911376003000981286212058382163224279001014840680
75	427119139069313849233653706011682481929449981218314	1414820907872080903089677397783217936935688462836496
76	320102703976116718596028190081374406273492525466550	1049603672627385446941931365449165555037423040587046
77	259995499018796919324183108088675668208737485025928	851192819749595103648102611913583622089201529554758
78	273637493819924114233834444570670545864028204184622	895893600793610096435375132209903318109897946571732
79	234664355295675582235329924172804223506767068427964	1316728660779894182529645086346895795302053663786410
80	898635262541893524532184602278401192465444239285926	3899466361833379305088464793170536393704331641986676
81	72867226173159815631008598164958377059688996878460	323324491961201814235688703776250808014832479341750
82	921193476257325623660129482010150258818800160074820	4506727486090015477534746747261793757779767270115004
83	51295885735525773756729065549038428620817375674408	296985412775528959346144424399224505274501929337434
84	105068470248592491399953616748982565162174848481114	353019455941183568903977461803755605507712028134126
85	72427943577307668053173328290121191922318065686152	249499239897060620429127302861028914365608253756416
86	16682845175158004242227594689665117819611933292074	73173668499644025944454062175573342154665625385780
87	625716465772011434176589731160867150416244664247234	2166289166691508865279455602158851142795264825040684
88	27898831055189742786517419474318361678994645780996	108623429714297552271812259948251489793337789158624
89	297742822948330466604234905697110966546920966168348	975337204113502755645444942684967721215722421689606
90	34155492715496155776260583844053570347249342254854	128402255997772349321041436010676994571685567732380
91	26617861669262943986346990231402849272889088638692	104574354818767276042305678382452727441372777043954
92	45340138746406652630570220026539497251472119518560	163769248713521110849490472700261098119129860688780
93	71381936381565077290856104832574225510301276621926	246186211841072281089843833432070865940936162168796
94	22837531466407204598598365755856789573654142377870	92625575107408800544354465769542497379464618056440

Table 12. Exact Banzhaf counts $\mathrm{Bz}_{\mathbf{i}}^{\mathbf{a}}$ for supermajority – part 3

i	2015	2016
95	4263395804249106673967134887144347832138694003353 0	15521071972573812782518229989525558319846134249411 8
96	247157849931994416381682310743110238430008662255444	80928486762033078109210377450357260788097570063431 0
97	54104934213516959447392068269135858247168076667212	36214182534160176510529879870866370294601101438403 2
98	103174011727540055347125360906146890295086611388140	34700283129575478603968822542668542555382911335246 8
99	29977613645414705823501642390509810402839894970228	16257508538694052635978234507943549627401494831594 0
100	41165262535925141133298445653237234427829732414728	15056621665808193785063975795508182560045157358162 4
101	30082596933358311863367988305279579489729633493986	16157993672519028428438684069018343347387037363697 4
102	372097158010579815000127938332207724258483127407344	122479556153970111232356028187950097517771817904297 4
103	17607176853345827147488825833336299748942951505408	7609491174447381731352526561812839932394541536897 2
104	35100102708042928775692200140549095075589909309816	13138869583581254601866259968146717794296121542377 6
105	36926195904961601760531427233595456033985298072736	13716224250088231372832712179748380509498862597049 0
106	16241678228828441634272693338718510465017703729840	7177942307831396416405542885538268831992241634453 4
107	29032738933494364740918493236106414801725559594172	11220775159434317754280382610053774577742603531678 8
108	36842242125814577711151762622104539697070809298124	16383559063636921245927795738829866790419970802257 6
109	658429980657095723921258525543388211278102103097830	409259940802694390609697092086350258196701622833196 8
110	16577805922400810990207817852748375389354435845510	7284170621016693812872210190363476979596136125306 8
111	41375085588727089510074638397283989746543395077070	15122973135225959981313164721971521467782172352117 2
112	26239856311583725206470232127005363201074091138932	10337952242474573905673136453798258446990082182796 6
113	21283229079515010587664322903939874931300920058392	10961908473290662967593829365007090132110094431110 2
114	138343302238119869978048535918562302137010974768126	65995414217073701903993067153221160906241533644817 0
115	39360658195688532995196642824874053361834805977878	14485979856662612103103958125886117756834114763949 4
116	69708046984250906606777176168622166913495672795212	41181297664009396060383695238532968362412512341948 0
117	44165430296283384904392902681317494433711947779290	16005402010914277003684955079865060175162652339864 2
118	30481528253731880881000945170476476615770037787560	11678756944831140825514997938021674576667102630238 4
119	801159861438024737416517543269521844015032064772354	406386662329208486362733692470107864543949372762640 4
120	201114117842684490278417219408809497948011344369306	66008489529945216232872988001926798803305821339866 8
121	42801799060759126182840715094062942405343405229092	15574150510387154840418831160106886749224975495900 6
122	29326704759345435683973225340648175464117818891984	11313700369781166045344583920456043353615680290953 6
123	368274638173735056913917836710487876810048105795966	121117836440034987022331865307056865121456618969260 16
124	392180606552639288993255744609657319581189966847726	129354350490059780697505820540170080034151099319246 8
125	65228768249789588889673498720251535803476383686434	22670181028896793069694796375967073018171896067292 4
126	229174407085256477945619422811656133098741261048682	138502093194624902968011532479990131563208049131869 8
127	16157645814331791333849248641747179993368332969400	7151385139617728526742782527687969396434106689217 8
128	58861860530298264163053163396010481058620644701860	20654825478423909718731107277919435578939450761255 2
129	43137474912216558714339366509099690464081735260782	15680306652188860987225493073726552087349331083131 0
130	36485433632751100629987616385109859502730995081522	13576865692529424988523918663745676338176123798378 0
131	148705573148532028859309465148436038627089860935586	9454127584860700513645014515441504768583709506139 08
132	226284067197627358806749755511022195365992348684772	7414435748338657459842995430518468115655309982572 82
133	356236123616371604143024891734638894690577545878134	254810420579178572117461086934356047465210751164645 4
134	228371885071732382747365381760424935292207457150878	140298681664313841016390158782934900443644746304976 4
135	78951880466284914921859406408387292642324886327146	27016804044557638700173598125313124817565103203711 4
136	228472215099762370912687084734720303503425913098116	124832695664867277822933746245080796498685975151827 6
137	844578514179715762286175337206338775684640321732952	334401552347541710854445793786778336710788523118819 0
138	32329100551005579465497455340422328020052242406472	12262824396346934298293886808700328829281129741146 8
139	17943291178624029062016441286002261014383879630090	7715717071157210208889471681855748094338092876763 2
140	20211969276414938200409960556047793461990022744336	10211829911847117713603827793756394757091468499036 0
141	17060983873868414331607994948401989966086037986618	7928167007196617240755837277150168674647959961242 4

TABLE 13. Exact Banzhaf counts $Bz_{\mathbf{i}}^{\mathbf{a}}$ for supermajority – part 4

i	2015	2016
142	880719880889746205050554576253671936253306729604848	3676265198068175422113349665153212813969097013199656
143	49471727597057335306131923257174325184458483338542	176837594941662819439217298459469758549843454999802
144	113367090641059940461002376473927867025289026259444	502769941764486824601335626776974734248420124375276
145	17796241582555972529647845427695103090190196272002	84659194321239810261578274676558081484774259150292
146	37282994499005042799257701148151378617750538909674	138290369392939768640092980732285176104943285779542
147	302950294620727927003732396031052197317326829457440	992581627260105769126385986812033471596368636275590
148	105006025900540763870469317668724012845234490381094	352821120109159596707036375327612661869380642957788
149	73180984693439693406200627034515880117061928197812	251884499830842143958844795432489463047717781259252
150	17691205754979234292178928303019155739528384162472	76360477059389231712683350684515845654749121565358
151	24790777357630579979483342479008641445722572585636	98799235827482065044783250670716175601137797471248
152	389420016880421449047933347880506266183875295159852	1284054262154075301744682497418517953005384573579774
153	41333121226469003395749040862986612005537642385676	151097028790162650767786073084080138043892572458962
154	703625551700633802084726982256878042484329842530966	4181657616848548563207176698724544125107249686795036
155	102070349966961765938279284905071954117453109851396	343498229809817036864437926031358425560108020217194
156	17376096284502694686549163684710986749876514974292	75364605173663322058106816030054645036028612283986
157	18720542216729127526972664706009320396059585114554	79613620826102780419109963595169781914717519234144
158	17250051670082639430907341724541186252217066309220	74966254934016033883719514175167987709340168795398
159	51128159079360974285208108210994464482278303275382	182077608333638137478019509689555293556453200663636
160	34848211875579916106917624780956215983343591640682	130592319916955844175406739342107995723421104165392
161	26155854249598213262701292724455230519543120787476	121566324765327431359648728539242718213267942870974
162	481155692953027402833353473174228066192983380530142	2711077519641822066086156241138279307208571731608578
163	637628919394053516604696883821764866212434498506142	3280222955914338215512289623169998048499945328489082
164	77070546649325347191726653595313148340944041062670	264206700203543797131414467008497367696168694353984
165	33777634078221453502967232664281089311705126543314	127207657244254074434305043051761718324570981813036
166	57248523770176951578619487804296985377590691066224	201442603541969913869834779977775421894493781004388
167	309220557035229307946192269328507701487898719208898	1013386451603492514808993883262971622376291759042564
168	17775234443743976280090214225778679596111889898336	76626041993094452670764630109376643121981759350686
169	30922442714569656094922886550009124379734917909896	118181392937341946740113947226879966544720598494002
170	16955945759502446396953175832810693075286723447726	74036767774542286259496180358083248325405985503994
171	85845911394652319434031843191384078174242647241428	292020337380272958999468015197140647717577649516026
172	75586052093849638107666891583565350584722758822410	259503393622165553103274100492036189860151479069272
173	312164219168035086597418004447033718209086727614410	1023169367115164421580272840672037686885225450315538
174	31300361240215936383810928286385435535977073098108	227762347842011204935818418693284013134794759786436
175	15884539129744374303611865979490952107271850497608	70650740970442997657136048561732914882735505454096
176	53392258183552849855544344919447653289312881881954	189240595463816273570767781540311263596929251699642
177	295977623071842167691253971024142813986026431224752	1373147646994403720957994798751302411527510718507796
178	172151003057887240432082635769627208313318698559722	566987984410544400059642434915453335938579197185028
179	918367775817962485090852104088421114577954611639702	4307308159804972843954035938800320763350232332137884
180	921227438210417921265982815126428343408780584986592	4506727722110247822679513808100007271801182981184082
181	79766976727463414558895487721181728250758031914148	353878899769397761584099801390661090976220099605868
182	73306484643173074040096084042196759208684701130908	252282033284318092722072486668181262568819094109082
183	19077651144993568321702222711101082031759486721940	80742248729502615043628226096705832123400145418278
184	523715593675375780846911480357914696688526319566394	1764192802606247735182311668928708934414293255783008
185	111912142122300910198074514742074381085150588252818	374764157905662003842361351824920708101981729995764
186	66589541128783380635395841608260717877252403267922	231010130636015903697304858513441780437951514163004
187	117812473763697246341738454407594170620364825779492	393524527557957632210276476880820633320151229597140
188	89561633679521016730626194579619932773406028812900	303803266060147410773398938449545501528813181752666