PARTIAL SPREADS AND VECTOR SPACE PARTITIONS

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ABSTRACT. Constant-dimension codes with the maximum possible minimum distance have been studied under the name of partial spreads in Finite Geometry for several decades. Not surprisingly, for this subclass typically the sharpest bounds on the maximal code size are known. The seminal works of Beutelspacher and Drake & Freeman on partial spreads date back to 1975, and 1979, respectively. From then until recently, there was almost no progress besides some computer-based constructions and classifications. It turns out that vector space partitions provide the appropriate theoretical framework and can be used to improve the long-standing bounds in quite a few cases. Here, we provide a historic account on partial spreads and an interpretation of the classical results from a modern perspective. To this end, we introduce all required methods from the theory of vector space partitions and Finite Geometry in a tutorial style. We guide the reader to the current frontiers of research in that field, including a detailed description of the recent improvements.

1. INTRODUCTION

Let $\mathbb{F}_q$ be the finite field with q elements, where $q > 1$ is a prime power. By $\mathbb{F}_q^v$ we denote the standard vector space of dimension $v \geq 1$ over $\mathbb{F}_q$, whose vectors are the v-tuples $\mathbf{x} = (x_1, \ldots, x_v)$ with $x_i \in \mathbb{F}_q$. The set of all subspaces of $\mathbb{F}_q^v$, ordered by the incidence relation $\subseteq$, is called *$(v-1)$-dimensional projective geometry over* $\mathbb{F}_q$ and denoted by $\operatorname{PG}(v-1, \mathbb{F}_q)$. It forms a finite modular geometric lattice with meet $X \wedge Y = X \cap Y$, join $X \vee Y = X + Y$, and rank function $X \mapsto \dim(X)$. Employing this algebraic notion of dimension instead of the geometric one, we will use the term *k-subspace* to denote a k-dimensional vector subspace of $\mathbb{F}_q^v$.[1] The important geometric interpretation of subspaces will still be visible in the terms *points, lines, planes, solids, hyperplanes* (denoting 1-, 2-, 3-, 4- and $(v-1)$-subspaces, respectively), and in general through our extensive use of geometric language.

In the same way as $\mathbb{F}_q^v$, an arbitrary v-dimensional vector space V over $\mathbb{F}_q$ gives rise to a projective geometry $\operatorname{PG}(V)$, and the terminology introduced before (and thereafter) applies to this general case as well. Since a vector space isomorphism $V \cong \mathbb{F}_q^v$ induces a geometric isomorphism ("collineation") $\operatorname{PG}(V) \cong \operatorname{PG}(\mathbb{F}_q^v) = \operatorname{PG}(v-1, \mathbb{F}_q)$, we could in principle avoid the use of non-standard vector spaces, but only at the expense of flexibility—for example, the Singer representation of the point-hyperplane design of $\operatorname{PG}(v-1, \mathbb{F}_q)$ is best developed using the field extension $\mathbb{F}_{q^v}/\mathbb{F}_q$ as ambient vector space V (and not $\mathbb{F}_q^v$, which would require a discussion of matrix representations of finite fields).

The set of all k-subspaces of an $\mathbb{F}_q$-vector space V will be denoted by $\begin{bmatrix} V \\ k \end{bmatrix}_q$. The sets $\begin{bmatrix} V \\ k \end{bmatrix}_q$ form finite analogues of the Graßmann varieties studied in Algebraic Geometry. In terms of $v = \dim(V)$, the cardinality of $\begin{bmatrix} V \\ k \end{bmatrix}_q$ is given by the Gaussian binomial coefficient

$$\begin{bmatrix} v \\ k \end{bmatrix}_q := \begin{cases} \frac{(q^v-1)(q^{v-1}-1)\cdots(q^{v-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} & \text{if } 0 \leq k \leq v; \\ 0 & \text{otherwise,} \end{cases}$$

which are polynomials of degree $k(v-k)$ in q (if they are nonzero) and represent q-analogues of the ordinary binomial coefficients in the sense that $\lim_{q \to 1} \begin{bmatrix} v \\ k \end{bmatrix}_q = \binom{k}{k}$. Their most important combinatorial properties are described in [2, Sect. 3.3] and [64, Ch. 24].

[1] Using the algebraic dimension has certain advantages—for example, the existence criterion $v = tk$ for spreads (cf. Theorem 1) looks ugly when stated in terms of the geometric dimensions: $v' = t(k'-1)+1$.

Making the connection with the main topic of this book, the geometry $\mathrm{PG}(v-1,\mathbb{F}_q)$ serves as input and output alphabet of the so-called *linear operator channel (LOC)*, a clever model for information transmission in coded packet networks subject to noise [42].[2] The relevant metrics on the LOC are given by the *subspace distance* $d_S(X,Y) := \dim(X+Y) - \dim(X\cap Y) = 2\cdot\dim(X+Y) - \dim(X) - \dim(Y)$, which can also be seen as the graph-theoretic distance in the Hasse diagram of $\mathrm{PG}(v-1,\mathbb{F}_q)$, and the *injection distance* $d_I(X,Y) := \max\{\dim(X),\dim(Y)\} - \dim(X\cap Y)$. A set $\mathscr{C}$ of subspaces of $\mathbb{F}_q^v$ is called a *subspace code* and serves as a channel code for the LOC in the same way as classical linear codes over $\mathbb{F}_q$ do for the q-ary symmetric channel.[3] The *minimum (subspace) distance* of $\mathscr{C}$ is given by $d = \min\{d_S(X,Y) \mid X,Y \in \mathscr{C}, X \neq Y\}$. If all elements of $\mathscr{C}$ have the same dimension, we call $\mathscr{C}$ a *constant-dimension code*. For a constant-dimension code $\mathscr{C}$ we have $d_S(X,Y) = 2d_I(X,Y)$ for all $X,Y\in\mathscr{C}$, so that we can restrict attention to the subspace distance. Constant-dimension codes are the most suitable for coding purposes, and the quest for good system performance leads straight to the problem of determining the maximum possible cardinality $\mathrm{A}_q(v,d;k)$ of a constant-dimension-k code in $\mathbb{F}_q^v$ with minimum subspace distance d. For two codewords X and Y of dimension k the inequality $d_S(X,Y)\geq d$ is equivalent to $\dim(X\cap Y)\leq k-d/2$.[4] Thus, the maximum possible minimum distance of a constant-dimension code with codewords of dimension k is $2k$. This extremal case has been studied under the name "partial spreads" in Finite Geometry for several decades. A *partial k-spread* in $\mathbb{F}_q^v$ is a collection of k-subspaces with pairwise trivial, i.e., zero-dimensional intersection. Translating this notion into Projective Geometry and identifying thereby, as usual, subspaces of $\mathbb{F}_q^v$ with their sets of incident points, we have that a partial k-spread in $\mathbb{F}_q^v$ is the same as a set of mutually disjoint k-subspaces, or $(k-1)$-dimensional flats in the geometric view, of the geometry $\mathrm{PG}(v-1,\mathbb{F}_q)$.[5]

With the history of partial spreads in mind, it comes as no surprise that the sharpest bounds on the maximal code sizes $\mathrm{A}_q(v,d;k)$ of constant-dimension codes are typically known for this special subclass. The primary goal of this survey is to collect all available information on the numbers $\mathrm{A}_q(v,2k;k)$ and present this information in an accessible way. Following standard practice in Finite Geometry, we will refer to partial k-spreads of size $\mathrm{A}_q(v,2k;k)$ as *maximal partial k-spreads*.[6]

In the case of a perfect packing, i.e., a partition of the point set of $\mathrm{PG}(v-1,\mathbb{F}_q)$, we speak of a *k-spread*. Partitions into subspaces of possibly different dimensions are equivalent to vector space partitions. A *vector space partition* $\mathscr{C}$ of $\mathbb{F}_q^v$ is a collection of nonzero subspaces with the property that every non-zero vector is contained in a unique member of $\mathscr{C}$. If $\mathscr{C}$ contains m_d subspaces of dimension d, then $\mathscr{C}$ is said to be of type $k^{m_k}\cdots 1^{m_1}$. Zero frequencies $m_d = 0$ are usually suppressed.[7] So, partial k-spreads are just the special case of vector space partitions, in which all members have dimension either k or 1. For $k\geq 2$ (the case $k=1$ is trivial) the members of dimension 1 correspond to points not covered by a k-subspace of the partial spread and are called *holes* in this context.

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[2]The use of distributed coding at the nodes of a packet-switched network, generally referred to as *Network Coding*, is described in [26, 54, 68] and elsewhere in this volume.

[3]except that attention is usually restricted to "one-shot subspace codes", i.e. subsets of the alphabet, which makes no sense in the classical case

[4]Note that the distance between codewords of the same dimension, and hence also the minimum distance of a constant-dimension code, is an even integer.

[5]In other words, partial spreads are just packings of the point set of a projective geometry $\mathrm{PG}(v-1,\mathbb{F}_q)$ into subspaces of equal dimension.

[6]The weaker property of *complete* (i.e., inclusion-maximal) partial spreads will not be considered.

[7]Since $\sum_{X\in\mathscr{C}}\dim(X) = \sum_d dm_d = v$, the type of $\mathscr{C}$ can be viewed as an ordinary integer partition of v.

Although vector space partitions can be seen as a mixed-dimension analogue of partial spreads, they are not usable as subspace codes of their own.[8] However, it turns out that they provide an appropriate framework to study bounds on the sizes of partial spreads.

There is a vast amount of related work that we will not cover in this survey: Partial spreads have also been studied for combinatorial designs and in polar spaces; for the latter see, e.g., [4, 20]. In the special case $v = 2k$ spreads can be used to define translation planes and provide a rich source for constructing non-desarguesian projective planes [40, 41, 49]. Also motivated by this geometric point-of-view, partial k-spreads in $\mathbb{F}_q^{2k}$ of size close to the maximum size (given by Theorem 1) have been studied extensively. Most of this research has focused on partial spread replacements and complete partial spreads, while we consider only partial spreads of maximum cardinality and hence do not touch the case $v = 2k$ (except for Theorem 1). The classification of all (maximal) partial spreads up to isomorphism, see e.g. [53], is also not treated here. Further, there is a steady stream of literature that characterizes the existing types of vector space partitions in $\mathbb{F}_2^v$ for small dimensions v. Here, we touch only briefly on some results that are independent of the ambient space dimension v and refer to [29] otherwise.

The remaining part of this chapter is structured as follows. In Section 2 we review some, mostly classical, bounds and constructions for partial spreads. After introducing the concept of q^r-divisible sets and codes in Section 3, we are able to obtain improved upper bounds for partial spreads in Theorems 9 and 10. Constructions for q^r-divisible sets are presented in Section 4, some non-existence results for q^r-divisible sets are presented in Section 5, and we close this survey with a collection of open research problems in Section 6.

2. Bounds and constructions for partial spreads

Counting points in $\mathbb{F}_q^v$ and $\mathbb{F}_q^k$ gives the obvious upper bound $\mathrm{A}_q(v, 2k; k) \le \begin{bmatrix} v \\ 1 \end{bmatrix}_q / \begin{bmatrix} k \\ 1 \end{bmatrix}_q = (q^v - 1) / \left(q^k - 1\right)$ for the size of a partial k-spread in $\mathbb{F}_q^v$. Equality corresponds to the case of spreads, for which a handy existence criterion is known from the work of Segre in 1964.[9]

Theorem 1 ([60, §VI], [16, p. 29])**.** *$\mathbb{F}_q^v$ contains a k-spread if and only if k is a divisor of v.*

Since $\frac{q^v-1}{q^k-1}$ is an integer if and only if k divides v (an elementary number theory exercise), only the constructive part needs to be shown. To this end we write $v = kt$ for a suitable integer t, take the ambient space V as the restriction ("field reduction") of $(\mathbb{F}_{q^k})^t$ to $\mathbb{F}_q$, which clearly has dimension v, and define the k-spread $\mathscr{S}$ in $V/\mathbb{F}_q$ as the set of 1-subspaces of $V/\mathbb{F}_{q^k}$. That $\mathscr{S}$ is indeed a k-spread, is easily verified: Each member of $\mathscr{S}$ has dimension k over $\mathbb{F}_q$; the members form a vector space partition of V (this property does not depend on the particular field of scalars); and the size $\begin{bmatrix} t \\ 1 \end{bmatrix}_{q^k} = \frac{q^v-1}{q^k-1}$ of $\mathscr{S}$ is as required.[10]

Example 1. *We consider the parameters $q = 3$, $v = 4$, and $k = 2$. Using canonical representatives in $\mathbb{F}_9 \simeq \mathbb{F}_3[x]/(x^2+1)$, the $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_9 = 10$ points in $\mathbb{F}_9^2$ are generated by*

$$\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\x\end{pmatrix}, \begin{pmatrix}1\\x+1\end{pmatrix}, \begin{pmatrix}1\\x+2\end{pmatrix}, \begin{pmatrix}1\\2x\end{pmatrix}, \begin{pmatrix}1\\2x+1\end{pmatrix}, \begin{pmatrix}1\\2x+2\end{pmatrix}.$$

The particular point $P = \mathbb{F}_9 \binom{1}{x+1} = \left\{ \binom{0}{0}, \binom{1}{x+1}, \binom{2}{2x+2}, \binom{x}{x+2}, \binom{x+1}{2x}, \binom{x+2}{1}, \binom{2x}{2x+1}, \binom{2x+1}{x}, \binom{2x+2}{2} \right\}$ on the projective line $\mathrm{PG}(1, \mathbb{F}_9)$ defines a 2-subspace of $\mathbb{F}_9^2/\mathbb{F}_3 \cong$

[8] The subspace distance $d_S(X,Y)$ depends not only on $\dim(X \cap Y)$ but also on $\dim(X)$ and $\dim(Y)$, which are not constant in this case.

[9] Segre in turn built to some extent on work of André, who had earlier considered the special case $v = 2k$ in his seminal paper on translation planes [1].

[10] Alternatively, the member of $\mathscr{S}$ containing a nonzero vector $\mathbf{x}$ is the k-subspace $\mathbb{F}_{q^k}\mathbf{x}$ of $V/\mathbb{F}_q$.

$\mathbb{F}_3^4$, *whose* 8 *associated points are* $\mathbb{F}_3\mathbf{x}$, $\mathbf{x} \in P$, $\mathbf{x} \neq \binom{0}{0}$; *and similarly for the other points of* $\mathrm{PG}(1,\mathbb{F}_9)$. *These ten* 2*-subspaces form the* 2*-spread* $\mathscr{S}$.

Using any $\mathbb{F}_3$*-isomorphism* $\mathbb{F}_9^2/\mathbb{F}_3 \cong \mathbb{F}_3^4$, *we can translate* $\mathscr{S}$ *into a* 2*-spread* $\mathscr{S}'$ *of the standard vector space* $\mathbb{F}_3^4$. *Taking, for example, coordinates with respect to the basis* $(1,x)$ *of* $\mathbb{F}_9/\mathbb{F}_3$ *and extending to* $\mathbb{F}_9^2$ *in the obvious way translates P into the* 2*-subspace of* $\mathbb{F}_3^4$ *with vectors*

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\2\\2\end{pmatrix},\begin{pmatrix}0\\1\\2\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\\2\end{pmatrix},\begin{pmatrix}2\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\2\\1\\2\end{pmatrix},\begin{pmatrix}1\\2\\0\\1\end{pmatrix},\begin{pmatrix}2\\2\\2\\0\end{pmatrix}.$$

The other members of $\mathscr{S}'$ *are obtained in the same way.*

From now on we assume that k does not divide v and write $v = tk + r$ with $1 \le r \le k-1$. Since the cases $t \in \{0,1\}$ are trivial ($\mathrm{A}_q(r,2k;k) = 0$ and $\mathrm{A}_q(k+r,2k;k) = 1$), we also assume $t \ge 2$. The stated upper bound then takes the form

$$\mathrm{A}_q(v,2k;k) \le \left\lfloor \frac{q^v-1}{q^k-1} \right\rfloor = \frac{q^{tk+r}-q^r}{q^k-1} + \left\lfloor \frac{q^r-1}{q^k-1} \right\rfloor = \sum_{s=0}^{t-1} q^{sk+r} = q^r \begin{bmatrix} t \\ 1 \end{bmatrix}_{q^k}. \qquad (1)$$

We also see from this computation that the number of holes of a partial k-spread is at least $\frac{q^v-1}{q-1} \bmod \frac{q^k-1}{q-1} = \frac{q^r-1}{q-1}$. However, as we will see later, this bound can be improved further.

In accordance with (1) we make the following definition (similar to that in [6]): The number σ defined by $\mathrm{A}_q(v;2k;k) = \sum_{s=0}^{t-1} q^{sk+r} - \sigma$ is called the *deficiency* of the maximal partial k-spreads in $\mathbb{F}_q^v$.[11] From (1) we have $\sigma \ge 0$. In terms of the deficiency, the minimum possible number of holes is $\sigma \cdot \frac{q^k-1}{q-1} + \frac{q^r-1}{q-1}$.

Our next goal is to derive a good lower bound for $\mathrm{A}_q(v,2k;k)$ (equivalently, a lower bound for the corresponding deficiency) by constructing a large partial k-spread in $\mathbb{F}_q^v$. For this we will employ a special case of the *echelon-Ferrers construction* for general subspace codes [23], which involves only standard maximum rank distance codes of full row rank.

To this end, recall that every k-subspace X of $\mathbb{F}_q^v$ is the row space of a unique "generating" matrix $\mathbf{A} \in \mathbb{F}_q^{k\times v}$ in reduced row-echelon form, which can be obtained by applying the Gaussian elimination algorithm to an arbitrary generating matrix of X. This matrix $\mathbf{A}$ is called *canonical matrix* of X, and is uniquely specified by its k pivot columns $1 \le j_1 < j_2 < \cdots < j_k \le v$ (forming a $k \times k$ identity submatrix of $\mathbf{A}$) and the complementary submatrix $\mathbf{B} \in \mathbb{F}_q^{k\times(v-k)}$, which has zero entries in positions (i,j) with $j \le j_i - i$ but otherwise can be arbitrary. The k-set $\{j_1,\dots,j_k\}$ will be named *pivot set* of X. The positions of the unrestricted entries in $\mathbf{B}$ form the Ferrers diagram of an integer partition, as shown in the following for the cases $v=8$, $k=3$, $(i_1,i_2,i_3) = (1,2,3)$, $(2,4,7)$.

matrix shape	Ferrers diagram	integer partition
$\begin{pmatrix}1&0&0&*&*&*&*&*\\0&1&0&*&*&*&*&*\\0&0&1&*&*&*&*&*\end{pmatrix}$	$\begin{array}{ccccc}\bullet&\bullet&\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet&\bullet&\bullet\end{array}$	$15 = 5+5+5$
$\begin{pmatrix}0&1&*&0&*&*&0&*\\0&0&0&1&*&*&0&*\\0&0&0&0&0&0&1&*\end{pmatrix}$	$\begin{array}{cccc}\bullet&\bullet&\bullet&\bullet\\&\bullet&\bullet&\bullet\\&&&\bullet\end{array}$	$8 = 4+3+1$

(2)

The following lemma is a special case of [23, Lemma 2].

Lemma 1. *If subspaces* X, Y *of* $\mathbb{F}_q^v$ *have disjoint pivot sets, they are itself disjoint (i.e.,* $X \cap Y = \{\mathbf{0}\}$*).*

[11] This makes sense also for $r = 0$: Spreads are assigned deficiency $\sigma = 0$.

Proof. A nonzero vector in X must have its pivot (first nonzero position) in the pivot set of X, and similarly for Y. The result follows. □ □

Now we focus on the special case in which the pivot set is $\{1,\dots,k\}$, i.e., the canonical matrix has the "systematic" form $\mathbf{A} = (\mathbf{I}_k|\mathbf{B})$. For matrices $\mathbf{A},\mathbf{B} \in \mathbb{F}_q^{k\times v}$ the *rank distance* is defined as $\mathrm{d_R}(\mathbf{A},\mathbf{B}) := \mathrm{rk}(\mathbf{A}-\mathbf{B})$. The subspace distance of two k-subspaces with pivot set $\{1,\dots,k\}$ can be computed from the rank distance of the corresponding canonical matrices:

Lemma 2 ([62, Prop. 4])**.** *Let X,X' be k-subspaces of $\mathbb{F}_q^v$ with canonical matrices $(\mathbf{I}_k|\mathbf{B})$ and $(\mathbf{I}_k|\mathbf{B}')$, respectively. Then* $\mathrm{d_S}(X,X') = 2\cdot \mathrm{d_R}(\mathbf{B},\mathbf{B}')$. [12]

Proof. The matrix $\begin{pmatrix}\mathbf{I}_k & \mathbf{B}\\ \mathbf{I}_k & \mathbf{B}'\end{pmatrix}$ generates $X+X'$ and reduces via Gaussian elimination to $\begin{pmatrix}\mathbf{I}_k & \mathbf{B}\\ \mathbf{0} & \mathbf{B}'-\mathbf{B}\end{pmatrix}$. Hence $\dim(X+X') = k+\mathrm{rk}(\mathbf{B}'-\mathbf{B}) = k+\mathrm{d_R}(\mathbf{B},\mathbf{B}')$ and $\mathrm{d_S}(X,X') = 2\dim(X+X')-2k = 2\,\mathrm{d_R}(\mathbf{B},\mathbf{B}')$. □ □

The so-called *lifting construction* [62, Sect. IV.A] associates with a matrix code $\mathscr{B} \subseteq \mathbb{F}_q^{k\times(v-k)}$ the constant-dimension code $\mathscr{C}$ in $\mathbb{F}_q^v$ whose codewords are the k-spaces generated by $(\mathbf{I}_k|\mathbf{B})$, $\mathbf{B}\in\mathscr{B}$. By Lemma 2, the code $\mathscr{C}$ is isometric to $\mathscr{B}$ with scale factor 2. In particular, $\mathscr{C}$ is a partial k-spread if and only if $\mathscr{B}$ has minimum rank distance $\mathrm{d_R}(\mathscr{B}) = k$.

Lemma 3. *There exists a partial k-spread $\mathscr{S}$ of size q^{v-k} in $\mathbb{F}_q^v$ whose codewords cover precisely the points outside the $(v-k)$-subspace $S = \{\mathbf{x}\in\mathbb{F}_q^v; x_1 = x_2 = \dots = x_k = 0\}$.*

Proof. Write $n = v-k$ and consider a matrix representation $M\colon \mathbb{F}_q^{n\times n}$ of $\mathbb{F}_{q^n}/\mathbb{F}_q$, obtained by expressing the multiplication maps $\mu_\alpha\colon \mathbb{F}_{q^n}\to\mathbb{F}_{q^n}$, $x\mapsto \alpha x$ (which are linear over $\mathbb{F}_q$) in terms of a fixed basis of $\mathbb{F}_{q^n}/\mathbb{F}_q$. Then $M(\alpha+\beta) = M(\alpha)+M(\beta)$, $M(\alpha\beta) = M(\alpha)M(\beta)$, $M(1) = \mathbf{I}_n$, and hence all matrices in $M(\mathbb{F}_{q^n})$ are invertible and have mutual rank distance n.[13]

Now let $\mathscr{B}\subseteq\mathbb{F}_q^{k\times n}$ be the matrix code obtained from $M(\mathbb{F}_{q^n})$ by deleting the last $n-k$ rows, say, of every matrix. Then $\#\mathscr{B} = q^n$ and $\mathrm{d_R}(\mathscr{B}) = k$. Hence by applying the lifting construction to $\mathscr{B}$ we obtain a partial k-spread $\mathscr{S}$ in $\mathbb{F}_q^v$ of size $q^n = q^{v-k}$ (Lemma 2).

The codewords in $\mathscr{S}$ cover only points outside S (compare the proof of Lemma 1). It remains to show that every such point is covered. This can be done by a counting argument or in the following more direct fashion: Let $\mathbf{a}\in\mathbb{F}_q^k\setminus\{\mathbf{0}\}$, $\mathbf{b}\in\mathbb{F}_q^{v-k}$ be arbitrary vectors and consider the equation $\mathbf{a}\mathbf{X} = \mathbf{b}$ for $\mathbf{X}\in\mathscr{B}$. Since $\mathrm{rk}(\mathbf{X}-\mathbf{X}') = k$ for $\mathbf{X}\neq\mathbf{X}'$, the q^{v-k} elements $\mathbf{a}\mathbf{X}$, $\mathbf{X}\in\mathscr{B}$, are distinct and hence account for all elements in $\mathbb{F}_q^{v-k}$. Thus the equation has a solution $\mathbf{B}\in\mathscr{B}$, and the point $P = \mathbb{F}_q(\mathbf{a}|\mathbf{b}) = \mathbb{F}_q\mathbf{a}(\mathbf{I}_k|\mathbf{B})$ is covered by the codeword in $\mathscr{S}$ with canonical matrix $(\mathbf{I}_k|\mathbf{B})$. □ □

Now we are ready for the promised construction of large partial k-spreads.

Theorem 2 ([6])**.** *Let v,k be positive integers satisfying $v = tk+r$, $t\geq 2$ and $1\leq r\leq k-1$. There exists a partial k-spread $\mathscr{S}$ in $\mathbb{F}_q^v$ of size*

$$\#\mathscr{S} = 1+\sum_{s=1}^{t-1} q^{v-sk} = 1+\sum_{s=1}^{t-1} q^{sk+r},$$

and hence we have $\mathrm{A}_q(v,2k;k)\geq 1+\sum_{s=1}^{t-1} q^{sk+r}$.

The corresponding bound for the deficiency is $\sigma\leq q^r-1$. It depends only on k and the residue $r = v \bmod k$.

[12] More generally, this formula holds if X and X' have the same pivot set and $\mathbf{B},\mathbf{B}'\in\mathbb{F}_q^{k\times(v-k)}$ denote the corresponding complementary submatrices in their canonical matrices; see e.g. [61, Corollary 3].

[13] In ring-theoretic terms, the matrices in $M(\mathbb{F}_{q^n})$ form a maximal subfield of the ring of $n\times n$ matrices over $\mathbb{F}_q$.

It had been conjectured in [20, Sect. 2.2] that $\sigma = q^r - 1$ in general, but this conjecture was later disproved in [21] by exhibiting a maximal partial plane spread of size 34 in $\mathbb{F}_2^8$, which has deficiency $2^2 - 2$.

Proof. The proof is by induction on t, using Lemma 2 and applying the inductive hypothesis to $S \cong \mathbb{F}_q^{v-k}$. The case $v = k + r$, in which $\mathrm{A}_q(v, 2k; k) = 1$, serves as the anchor of the induction. □ □

The partial spread $\mathscr{S}$ exhibited in the proof of Theorem 2 consists of $t-1$ "layers" $\mathscr{S}_1$, …, $\mathscr{S}_{t-1}$ of decreasing sizes $\#\mathscr{S}_s = q^{v-sk}$, whose codewords are obtained from matrix representations of $\mathbb{F}_{q^{v-sk}}$ and have their pivots in positions $(s-1)k+1$, $(s-1)k+2$, …, sk (hence vanish on the first $(s-1)k$ coordinates). The union $\bigcup_{s=1}^{t-1} \mathscr{S}_s$ leaves exactly the points of a $(k+r)$-subspace S of $\mathbb{F}_q^v$ (the span of the last $k+r$ standard unit vectors) uncovered. Finally, one further k-subspace S_0 of S is selected to form $\mathscr{S} = \bigcup_{s=1}^{t-1} \mathscr{S}_s \cup \{S_0\}$.[14]

Example 2. *We consider the particular case $q = 2$, $v = 5$, $k = 3$, in which $\#\mathscr{S} = 2^3 + 1 = 9$. In this case there is only one layer $\mathscr{S}_1$, which can be obtained from a matrix representation of $\mathbb{F}_8$ as follows: Representing $\mathbb{F}_8$ as $\mathbb{F}_2(\alpha)$ with $\alpha^3 + \alpha + 1 = 0$, we first express the powers α^j, $0 \le j \le 6$ in terms of the basis $1, \alpha, \alpha^2$ of $\mathbb{F}_8/\mathbb{F}_2$, as in the following matrix:*

$$\mathbf{M} = \begin{array}{c||ccccccc|cc} & \alpha^0 & \alpha^1 & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 & \alpha^0 & \alpha^1 \\ \hline\hline \alpha^0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ \alpha^1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ \alpha^2 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{array} \tag{3}$$

The seven consecutive 3×3 submatrices of this matrix, which has been extended to the right in order to mimic the cyclic wrap-around, form a matrix field isomorphic to $\mathbb{F}_8$ (with $\mathbf{0} \in \mathbb{F}_2^{3\times 3}$ added). Similarly, the code $\mathscr{B} \subset \mathbb{F}_2^{2\times 3}$ is obtained by extracting the first seven consecutive 3×2 submatrices, adding $\mathbf{0} \in \mathbb{F}_2^{3\times 2}$, and transposing; cf. the proof of Lemma 3). Prepending the 2×2 identity matrix then gives the canonical matrices of the 8 codewords of $\mathscr{S}_1$:

$$\left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right), \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right),$$
$$\left(\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right), \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right), \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right), \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right).$$

Finally, from the seven lines in the plane $S = \{\mathbf{x} \in \mathbb{F}_2^5; x_1 = x_2 = 0\}$ a 9th codeword L_0 (moving line) is selected to form $\mathscr{S} = \mathscr{S}_1 \cup \{L_0\}$.

The partial line spread $\mathscr{S}$ is in fact maximal, as we will see in a moment, and represents one of the 4 isomorphism types of maximal partial line spreads in $\mathrm{PG}(\mathbb{F}_2^5) = \mathrm{PG}(4, \mathbb{F}_2)$.[15]

Now we will reduce the upper bound (1) by a summand of $q-1$, which is sufficient to settle the case $r = 1$ and hence determine the numbers $\mathrm{A}_q(tk+1, 2k; k)$. The key ingredient will be the observation that a partial k-spread induces in every hyperplane a vector space partition, whose members have dimension k, $k-1$, or 1. Before turning to the general case, which is a little technical, we continue the preceding example and illustrate the method for partial line spreads in $\mathbb{F}_2^5$.

The geometry $\mathrm{PG}(4, \mathbb{F}_2)$ has 31 points, each line containing 3 points, and thus it is conceivable that a partial line spread $\mathscr{S}$ of size 10 exists in $\mathrm{PG}(4, \mathbb{F}_2)$. But in fact it does

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[14] The space S_0 has been named *moving subspace* of $\mathscr{S}$, since it can be freely "moved" within S without affecting the partial spread property of $\mathscr{S}$.

[15] The full classification, including also partial line spreads of smaller size, can be found in [25]. To our best knowledge there is only one further nontrivial parameter case, where a classification of maximal (proper) partial spreads is known, viz. the case of plane spreads in $\mathrm{PG}(6, \mathbb{F}_2)$, settled in [36].

not. To prove this, consider a hyperplane (solid) H in $\mathrm{PG}(4,\mathbb{F}_2)$. If H contains α lines of $\mathscr{S}$, it meets the remaining $\#\mathscr{S}-\alpha$ lines in a point, giving the constraint $\alpha\cdot 3+(\#\mathscr{S}-\alpha)\cdot 1\leq 15$, the total number of points in H. This is equivalent to $\#\mathscr{S}\leq 15-2\alpha$. In order to complete the proof, we need to show that there exists a hyperplane containing at least 3 lines of $\mathscr{S}$. This can be done by an averaging argument. On average, a hyperplane contains

$$\sum_{H}\frac{\#\{L\in\mathscr{S};L\subset H\}}{2^5-1}=\sum_{L\in\mathscr{S}}\frac{\#\{H;H\supset L\}}{2^5-1}=\frac{\#\mathscr{S}(2^3-1)}{2^5-1}=\frac{7}{31}\cdot\#\mathscr{S} \tag{4}$$

lines of $\mathscr{S}$. If $\#\mathscr{S}\geq 9$, this number is >2, implying the desired conclusion $\sigma\geq 1$.[16]

The general case is the subject of the following

Theorem 3 ([20, Th. 2.7(a)]). *The deficiency of a maximal k-spread in $\mathbb{F}_q^v$, where k does not divide v, is at least $q-1$.*

Proof. Reasoning as in the preceding example gives $\alpha\cdot\frac{q^k-1}{q-1}+(\#\mathscr{S}-\alpha)\frac{q^{k-1}-1}{q-1}\leq\frac{q^{tk+r-1}-1}{q-1}$ and hence the bound

$$\#\mathscr{S}\leq\frac{q^{tk+r-1}-1-\alpha(q^k-q^{k-1})}{q^{k-1}-1} \tag{5}$$

for any partial k-spread $\mathscr{S}$ having a hyperplane incident with α members of $\mathscr{S}$.

Now suppose $\#\mathscr{S}=1+\sum_{s=1}^{t-1}q^{sk+r}$, the same size as the partial k-spread in Theorem 2. In this case the average number of codewords contained in a hyperplane is

$$\begin{aligned}\frac{q^{(t-1)k+r}-1}{q^{tk+r}-1}\left(1+\sum_{s=1}^{t-1}q^{sk+r}\right)&=\frac{1}{q^{tk+r}-1}\left(\sum_{s=t}^{2t-2}q^{sk+2r}-\sum_{s=1}^{t-2}q^{sk+r}-1\right)\\&=\frac{1}{q^{tk+r}-1}\left(\sum_{s=t}^{2t-2}q^{sk+2r}-\sum_{s=0}^{t-2}q^{sk+r}+q^r-1\right)\\&=\sum_{s=0}^{t-2}q^{sk+r}+\frac{q^r-1}{q^{tk+r}-1}.\end{aligned}$$

It follows that $\mathscr{S}$, and likewise all partial k-spreads of deficiency $\leq q^r-1$, have a hyperplane containing at least $1+\sum_{s=0}^{t-2}q^{sk+r}$ codewords. Substituting this number into (5) gives

$$\begin{aligned}\#\mathscr{S}&\leq\frac{1}{q^{k-1}-1}\left(q^{tk+r-1}-1-\left(1+\sum_{s=0}^{t-2}q^{sk+r}\right)(q^k-q^{k-1})\right)\\&=\frac{1}{q^{k-1}-1}\left(q^{tk+r-1}-1-q^k+q^{k-1}-\sum_{s=1}^{t-1}q^{sk+r}+\sum_{s=0}^{t-2}q^{sk+r+k-1}\right)\\&=1+\sum_{s=1}^{t-2}q^{sk+r}+\frac{q^{tk+r-1}-q^k-q^{(t-1)k+r}+q^{r+k-1}}{q^{k-1}-1}\\&=1+\sum_{s=1}^{t-1}q^{sk+r}+\frac{q^{r+k-1}-q^k}{q^{k-1}-1}\\&=1+\sum_{s=1}^{t-1}q^{sk+r}+q^r-q+\frac{q^r-q}{q^{k-1}-1}\\&=\sum_{s=0}^{t-1}q^{sk+r}-(q-1)+\frac{q^r-q}{q^{k-1}-1},\end{aligned}$$

[16] In this particular case one may also argue as follows: If $\#\mathscr{S}=10$ then there is only one hole and the hyperplane constraint becomes $3\alpha+(10-\alpha)+h=15$, where $h\in\{0,1\}$. This forces $\alpha=2$ and $h=1$, i.e., every hyperplane should contain the hole. This is absurd, of course.

valid now for any partial k-spread $\mathscr{S}$ in $\mathbb{F}_q^v$. Since the last summand is <1, we obtain the desired conclusion $\sigma \geq q-1$. □ □

Theorem 3 has the following immediate corollary, established by Beutelspacher in 1975, which settles the case $r = 1$ completely.

Corollary 1 ([6, Th. 4.1]; see also [34] for the special case $q = 2$). *For integers $k \geq 2$ and $v = tk+1$ with $t \geq 1$ we have $\mathrm{A}_q(v, 2k; k) = 1 + \sum_{s=1}^{t-1} q^{sk+1}$,*[17] *with corresponding deficiency $\sigma = q-1$.*[18]

In particular, maximal partial line spreads in $\mathbb{F}_q^v$, v odd (the case where no line spreads exist), have size $q^{v-2} + q^{v-4} + \cdots + q^3 + 1$, deficiency $q-1$, and q^2 holes.

In his original proof of the corollary Beutelspacher considered the set of holes N and the average number of holes per hyperplane, which is less than the total number of holes divided by q. An important insight was the relation $\#N \equiv \#(H \cap N) \pmod{q^{k-1}}$ for each hyperplane H, i.e., the number of holes per hyperplane satisfies a certain modulo constraint. We will see this concept in full generality in Section 3. In terms of integer linear programming, the upper bound is obtained by an integer rounding cut. The construction in [6, Theorem 4.2] recursively uses arbitrary k'-spreads, so that it is more general than the one of Theorem 2.

For a long time the best known upper bound on $\mathrm{A}_q(v, 2k; k)$, i.e., the best known lower bound on σ, was the one obtained by Drake and Freeman in 1979:

Theorem 4 (Corollary 8 in [19]). *The deficiency of a maximal partial k-spread in $\mathbb{F}_q^v$ is at least $\lfloor \theta \rfloor + 1 = \lceil \theta \rceil$,*[19] *where $2\theta = \sqrt{1 + 4q^k(q^k - q^r)} - (2q^k - 2q^r + 1)$.*

The authors concluded from the existence of a partial spread the existence of a (group constructible) (s, r, μ)-net and applied [10, Theorem 1B]—a necessary existence criterion formulated for orthogonal arrays of strength 2 by Bose and Bush in 1952. The underlying proof technique can be further traced back to [58] and is strongly related to the classical second-order Bonferroni Inequality [9, 24]; see also [37, Section 2.5] for an application to bounds for subspace codes.

Given Theorem 1 and Corollary 1, the first open binary case is $\mathrm{A}_2(8, 6; 3)$. The construction from Theorem 2 gives a partial spread of cardinality 33, while Theorem 4 implies an upper bound of 34. As already mentioned, in 2010 El-Zanati et al. [21] found a sporadic partial plane spread in $\mathbb{F}_2^8$ of cardinality 34 by a computer search. Together with the following easy lemma, this completely answers the situation for partial plane spreads in $\mathbb{F}_2^v$; see Corollary 2 below.

Lemma 4. *For fixed q, k and r the deficiency σ is a non-increasing function of $v = kt + r$.*

Proof. Let $\mathscr{S}$ be a maximal partial k-spread in $\mathbb{F}_q^{tk+r}$ and σ its deficiency, so that $\mathrm{A}_q(tk + r, 2k; k) = \sum_{s=0}^{t-1} q^{sk+r} - \sigma$. We can embed $\mathscr{S}$ into $\mathbb{F}_q^{(t+1)k+r}$ by prepending k zeros to each codeword. Then Lemma 3 can be applied and yields a partial k-spread $\mathscr{S}'$ in $\mathbb{F}_q^{(t+1)k+r}$ of size q^{tk+r}, whose codewords are disjoint from those in $\mathscr{S}$. This implies $\mathrm{A}_q((t+1)k + r, 2k; k) \geq \#\mathscr{S} \cup \mathscr{S}' = \sum_{s=0}^{t} q^{sk+r} - \sigma$, and hence the deficiency σ' of a maximal partial k-spread in $\mathbb{F}_q^{(t+1)k+r}$ satisfies $\sigma' \leq \sigma$. □ □

So, any improvement of the best known lower bound for a single parameter case gives rise to an infinite series of improved lower bounds. Unfortunately, so far, the sporadic

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[17] This can also written as $\mathrm{A}_q(v, 2k; k) = q^1 \cdot \frac{q^{v-1}-1}{q^k-1} - q + 1 = \frac{q^v - q^{k+1} + q^k - 1}{q^k - 1}$.

[18] The corresponding number of holes is q^k.

[19] Assuming $1 + 4q^k(q^k - q^r) = 1 + 4q^{k+r}(q^{k-r} - 1) = (2z-1)^2 = 1 + 4z(z-1)$ for some integer $z > 1$ implies $q^{k+r} \mid z$ or $q^{k+r} \mid z - 1$, so that $z \geq q^{k+r}$, which is impossible for $(k, r) \neq (1, 0)$. Thus, $2\theta \notin \mathbb{Z}$, so that $\theta \notin \mathbb{Z}$ and $\lfloor \theta \rfloor + 1 = \lceil \theta \rceil$.

construction in [21] is the only known example being strictly superior to the general construction of Theorem 2.

Corollary 2. *For each integer $m \geq 2$ we have* $\mathrm{A}_2(3m,6;3) = \frac{2^{3m}-1}{7}$, $\mathrm{A}_2(3m+1,6;3) = \frac{2^{3m+1}-9}{7}$, *and* $\mathrm{A}_2(3m+2,6;3) = \frac{2^{3m+2}-18}{7}$. *The corresponding deficiencies are* 0, 1 *and* 2, *respectively.*

Very recently, the case $q = r = 2$ was completely settled. For $k = 3$ the answer is given in the preceding corollary, and for $k \geq 4$ by the following

Theorem 5 ([45, Theorem 5])**.** *For integers $k \geq 4$ and $v = tk+2$ with $t \geq 1$ we have $\sigma = 3$ and* $\mathrm{A}_2(kt+2,2k;k) = 1+\sum_{s=1}^{t-1} 2^{tk+2} = \frac{2^{kt+2}-3\cdot 2^k-1}{2^k-1}$.[20]

The technique used to prove this theorem is very similar to the one presented in the proof of Theorem 3.

Corollary 3. *We have* $\mathrm{A}_2(4m,8;4) = \frac{2^{4m}-1}{15}$. $\mathrm{A}_2(4m+1,8;4) = \frac{2^{4m+1}-17}{15}$, $\mathrm{A}_2(4m+2,8;4) = \frac{2^{4m+2}-49}{15}$, *and* $\frac{2^{4m+3}-113}{15} \leq \mathrm{A}_2(4m+3,8;4) \leq \frac{2^{4m+3}-53}{15}$ *for all $m \geq 2$. The corresponding deficiencies are* 0, 1, 3 *and* $3 \leq \sigma \leq 7$, *respectively*

As a consequence, the first unknown binary case is now $129 \leq \mathrm{A}_2(11,8;4) \leq 133$.[21] For $r = 2$ and $q = 3$ the upper bound of Theorem 4 has been decreased by 1:

Lemma 5 (cf. [45, Lemma 4])**.** *For integers $t \geq 2$ and $k \geq 4$, we have $\sigma \geq 5$ and* $\mathrm{A}_3(kt+2,2k;k) \leq \frac{3^{kt+2}-3^2}{3^k-1} - 5$.

Again, the proof technique is very similar to that used in the proof of Theorem 3.

Theorem 2 is asymptotically optimal for $k \gg r = v \bmod k$, as recently shown by Năstase and Sissokho:

Theorem 6 ([57, Theorem 5])**.** *If $k > \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ then $\sigma = q^r - 1$ and* $\mathrm{A}_q(v,2k;k) = 1+\sum_{s=1}^{t-1} q^{sk+r}$.[22]

Choosing $q = r = 2$, this result covers Theorem 5. The same authors have refined their analysis, additionally using Theorem 14 from the theory of vector space partitions, to obtain improved upper bounds for some of the cases $k \leq \begin{bmatrix} r \\ 1 \end{bmatrix}_q$, see [56, Theorem 6 and 7]. Using the theory of q^r-divisible codes, presented in the next section, we extend their results further in Corollary 7 and Theorem 10.

3. q^r-DIVISIBLE SETS AND CODES

The currently most effective approach to good upper bounds for partial spreads follows the original idea of Beutelspacher and considers the set of holes as a stand-alone object. As it appears in the proof of Beutelspacher, the number of holes in a hyperplane satisfies a certain modulo constraint. In this section we consider sets of points in $\mathrm{PG}(v-1,\mathbb{F}_q)$ having the property that modulo some integer $\Delta > 1$ the number of points in each hyperplane is the same. Such point sets are equivalent to Δ-divisible codes [66, 67] with projectively distinct coordinate functionals (so-called *projective* codes), and this additional restriction forces Δ to be a power of the same prime as q. Writing $q = p^e$, p prime, and $\Delta = p^f$, we have $\Delta = q^r$ with $r = f/e \in \frac{1}{e}\mathbb{Z}$.

We will derive several important properties of these *q^r-divisible sets* and *codes* and in particular observe that the set of holes of a partial spread is exactly of this type. Without the notion of q^r-divisible sets and the reference to the linear programming method, almost all results of this section are contained in [46]. A more extensive introduction to

[20] Thus in all these cases $\sigma = 2^2 - 1$ and the partial spreads of Theorem 2 are maximal. This notably differs from the case $k = 3$.

[21] The upper bound can be sharpened to 132, as we will see later.

[22] This corresponds again to the upper bound $\sigma = q^r - 1$.

the topic, including constructions and relations to other combinatorial objects, is currently in preparation [30].

In what follows, we denote the point set of $\mathrm{PG}(v-1,\mathbb{F}_q)$ by $\mathscr{P}$ and call for subsets $\mathscr{C}\subseteq\mathscr{P}$ and subspaces X of $\mathbb{F}_q^v$ the integer $\#(\mathscr{C}\cap X)=\#\{P\in\mathscr{C};P\subseteq X\}$ the *multiplicity* of X with respect to $\mathscr{C}$.

Definition 1. *Let $\Delta>1$ be an integer. A set $\mathscr{C}$ of points in* $\mathrm{PG}(v-1,\mathbb{F}_q)$ *is called* weakly Δ-divisible *if there exists $u\in\mathbb{Z}$ with $\#(\mathscr{C}\cap H)\equiv u \pmod{\Delta}$ for each hyperplane H of* $\mathrm{PG}(v-1,\mathbb{F}_q)$*. If $u\equiv\#\mathscr{C} \pmod{\Delta}$, we call $\mathscr{C}$* (strongly) Δ-divisible.

Trivial cases are $\mathscr{C}=\emptyset$ (strongly Δ-divisible for any Δ) and $\mathscr{C}=\mathscr{P}$ (weakly Δ-divisible for any Δ, with largest strong divisor $\Delta=q^{v-1}$).[23]

It is well-known (see, e.g., [63, 17, Prop. 1]) that the relation $C\to\mathscr{C}$, associating with a full-length linear $[n,v]$ code C over $\mathbb{F}_q$ the n-multiset $\mathscr{C}$ of points in $\mathrm{PG}(v-1,\mathbb{F}_q)$ defined by the columns of any generator matrix, induces a one-to-one correspondence between classes of (semi-)linearly equivalent spanning multisets and classes of (semi-)monomially equivalent full-length linear codes. Point sets correspond in this way to projective linear codes, which are also characterized by the condition $\mathrm{d}(C^\perp)\geq 3$. The importance of the correspondence lies in the fact that it relates coding-theoretic properties of C to geometric or combinatorial properties of $\mathscr{C}$. An example is the formula

$$\mathrm{w}(\mathbf{aG})=n-\#\{1\leq j\leq n;\mathbf{a}\cdot\mathbf{g}_j=0\}=n-\#(\mathscr{C}\cap\mathbf{a}^\perp),\tag{6}$$

where w denotes the Hamming weight, $\mathbf{G}=(\mathbf{g}_1|\ldots|\mathbf{g}_n)\in\mathbb{F}_q^{v\times n}$ a generating matrix of C, $\mathbf{a}\cdot\mathbf{b}=a_1b_1+\cdots+a_vb_v$, and $\mathbf{a}^\perp$ is the hyperplane in $\mathrm{PG}(v-1,\mathbb{F}_q)$ with equation $a_1x_1+\cdots+a_vx_v=0$.

A linear code C is said to be Δ-*divisible* ($\Delta\in\mathbb{Z}_{>1}$) if all nonzero codeword weights are multiples of Δ. Following the Gleason-Pierce-Ward Theorem on the divisibility of self-dual codes (see, e.g., [39, Ch. 9.1]), a rich theory of divisible codes has been developed over time, mostly by H. N. Ward; cf. his survey [67]. One of Ward's results implies that nontrivial weakly Δ-divisible point sets in $\mathrm{PG}(v-1,\mathbb{F}_q)$ are strongly Δ divisible and exist only in the case $\Delta=p^f$. The proof uses the so-called *standard equations* for the hyperplane spectrum of $\mathscr{C}$, which we state in the following lemma. The standard equations are equivalent to the first three MacWilliams identities for the weight enumerators of C and $C^\perp$ (stated as Equation (11) below), specialized to the case of projective linear codes. The geometric formulation, however, seems more in line with the rest of the paper.

Lemma 6. *Let $\mathscr{C}$ be a set of points in* $\mathrm{PG}(v-1,\mathbb{F}_q)$ *with $\#\mathscr{C}=n$, and let a_i hyperplanes of* $\mathrm{PG}(v-1,\mathbb{F}_q)$ *contain exactly i points of $\mathscr{C}$ ($0\leq i\leq n$). Then we have*

$$\sum_{i=0}^{n}a_i = \begin{bmatrix}v\\1\end{bmatrix}_q,\tag{7}$$

$$\sum_{i=1}^{n}ia_i = n\cdot\begin{bmatrix}v-1\\1\end{bmatrix}_q,\tag{8}$$

$$\sum_{i=2}^{n}\binom{i}{2}a_i = \binom{n}{2}\cdot\begin{bmatrix}v-2\\1\end{bmatrix}_q.\text{[24]}\tag{9}$$

Proof. Double-count incidences of the tuples (H), (P_1,H), and $(\{P_1,P_2\},H)$, where H is a hyperplane and $P_1\neq P_2$ are points contained in H. □ □

[23] This is also true for $v=1$, where $\mathscr{C}=\emptyset,\mathscr{P}$ exhausts all possibilities.

[24] The general (multiset) version of (9) has an additional summand of $q^{v-2}\cdot\sum_{P\in\mathscr{P}}\binom{\mathscr{C}(P)}{2}$ on the right-hand side, accounting for the fact that "pairs of equal points" are contained in $\begin{bmatrix}v-1\\1\end{bmatrix}_q$ hyperplanes.

In the proof of Theorem 7 we will need that (7) and (8) remain true for any multiset $\mathscr{C}$ of points in $\mathrm{PG}(v-1,\mathbb{F}_q)$, provided points are counted with their multiplicities in $\mathscr{C}$ and the cardinality $\#\mathscr{C}$ is defined in the obvious way. We will also need the following concept of a *quotient multiset*. Let $\mathscr{C}$ be a set of points in $\mathrm{PG}(v-1,\mathbb{F}_q)$ and X a subspace of $\mathbb{F}_q^v$. Define the multiset $\mathscr{C}/X$ of points in the quotient geometry $\mathrm{PG}(\mathbb{F}_q^v/X)$ by assigning to a point Y/X of $\mathrm{PG}(\mathbb{F}_q^v/X)$ (i.e., Y satisfies $\dim(Y/X)=1$) the difference $\#(\mathscr{C}\cap Y)-\#(\mathscr{C}\cap X)=\#(\mathscr{C}\cap Y\setminus X)$ as multiplicity.[25] With this definition it is obvious that $\#(\mathscr{C}/X)=\#\mathscr{C}-\#(\mathscr{C}\cap X)$. In particular, if $\mathscr{C}$ is an n-set and $X=P$ is a point then $\#(\mathscr{C}/P)=n-1$ or n, according to whether $P\in\mathscr{C}$ or $P\notin\mathscr{C}$, respectively.[26]

Theorem 7. *Let $\mathscr{C}\neq\emptyset,\mathscr{P}$ be a weakly Δ-divisible point set in $\mathrm{PG}(v-1,\mathbb{F}_q)$, $n=\#\mathscr{C}$, and C any linear $[n,v]$-code over $\mathbb{F}_q$ associated with $\mathscr{C}$ as described above.*[27] *Then*
(i) $\mathscr{C}$ is strongly Δ-divisible;
(ii) C is Δ-divisible;
(iii) Δ is a divisor of q^{v-2}.

Proof. (i) and (ii) are equivalent in view of (6).

First we prove (i). Let u be as in Definition 1. Choose a point $P\notin\mathscr{C}$ and let $\mathscr{C}'=\mathscr{C}/P$. Then $\mathscr{C}'$ is an n-multiset of points in $\mathrm{PG}(\mathbb{F}_q^v/P)\cong\mathrm{PG}(v-2,\mathbb{F}_q)$ with $\#(\mathscr{C}'\cap H')\equiv u\pmod{\Delta}$ for each hyperplane H' of $\mathrm{PG}(\mathbb{F}_q^v/P)$. The hyperplane spectrum (a_i') of $\mathscr{C}'$ satisfies (8) with v replaced by $v-1$. Multiplying the identity for $\mathscr{C}'$ by q and subtracting from it the identity for $\mathscr{C}$ gives

$$\sum_{i\geq 0}(u+i\Delta)(a_{u+i\Delta}-qa'_{u+i\Delta})=n\left(\frac{q^{v-1}-1}{q-1}-\frac{q^{v-1}-q}{q-1}\right)=n.$$

Reading this equation modulo Δ and using (7) further gives

$$n\equiv u\sum_{i\geq 0}(a_{u+i\Delta}-qa'_{u+i\Delta})=u\left(\frac{q^{v}-1}{q-1}-\frac{q^{v}-q}{q-1}\right)=u\pmod{\Delta},$$

as desired. Thus (i) and (ii) hold.

For the proof of (iii) we use a point $Q\in\mathscr{C}$ and its associated quotient multiset $\mathscr{C}''=\mathscr{C}/Q$, which satisfies $\#\mathscr{C}''=n-1$ and $\#(\mathscr{C}''\cap H'')\equiv u-1\pmod{\Delta}$ for each hyperplane H'' of $\mathrm{PG}(\mathbb{F}_q^v/Q)$. Subtracting (8) for $\mathscr{C}'$ and $\mathscr{C}''$ gives

$$\sum_{i\geq 0}(u+i\Delta)a'_{u+i\Delta}-\sum_{i\geq 0}(u-1+i\Delta)a''_{u-1+i\Delta}=\frac{q^{v-2}-1}{q-1}.$$

Again reading the equation modulo Δ and using (7) gives

$$\frac{q^{v-2}-1}{q-1}\equiv u\sum_{i\geq 0}a'_{u+i\Delta}-(u-1)\sum_{i\geq 0}a''_{u-1+i\Delta}=\frac{q^{v-1}-1}{q-1}\pmod{\Delta}$$

or $q^{v-2}\equiv 0\pmod{\Delta}$, as asserted. □ □

Let us remark that Part (ii) of Theorem 7 also follows from [66, Th. 3], which asserts that a not necessarily projective code C satisfying the assumption of the theorem must be either Δ-divisible or the juxtaposition of a Δ-divisible code and a v-dimensional linear constant weight code. Since the latter is necessarily a repetition of simplex codes, this case

[25] This definition can be extended to multisets $\mathscr{C}$ by defining the multiplicity of Y/X in $\mathscr{C}/X$ as the sum of the multiplicities in $\mathscr{C}$ of all points in $Y\setminus X$.

[26] If $C\leftrightarrow\mathscr{C}$ then the multisets $\mathscr{C}/P$, $P\in\mathscr{P}$, are associated to the $(v-1)$-dimensional subcodes $D\subset C$, and the n points $P\in\mathscr{C}$ correspond to the n subcodes D of effective length $n-1$ ("D is C shortened at P"). This correspondence between points and subcodes extends to a correlation between $\mathrm{PG}(v-1,\mathbb{F}_q)$ and $\mathrm{PG}(C/\mathbb{F}_q)$, which includes the familiar correspondence between hyperplanes and codewords as a special case; see [63, 17] for details.

[27] It is not required that $\mathscr{C}$ is spanning; if it is not then (iii) sharpens to "Δ is a divisor of $q^{\dim\langle\mathscr{C}\rangle-2}$".

does not occur for projective codes. Our proofs of (i), (iii) use the very same ideas as in [66], translated into the geometric framework.[28]

Part (iii) of Theorem 7 says that exactly Δ-divisible point sets in $\mathrm{PG}(v-1,\mathbb{F}_q)$ exist only if $\Delta = q^r$ with $r \in \frac{1}{e}\mathbb{Z}$ and $r \leq v-1$;[29] the whole point set $\mathscr{P}$ has $\Delta = q^{v-1}$, and $v-2 < r < v-1$ does not occur. Conversely, it is not difficult to see that every divisor $\Delta > 1$ of q^{v-2} is the largest divisor of some point set in $\mathrm{PG}(v-1,\mathbb{F}_q)$.[30]

In the proof of Theorem 7 we have used that the (weak) divisibility properties of $\mathscr{C}$ and its quotient multisets $\mathscr{C}/X$ are the same. Now we consider the restrictions $\mathscr{C}\cap X$, which correspond to residual codes of the associated code C.

Lemma 7. *Suppose that $\mathscr{C}$ is a q^r-divisible set of points in $\mathrm{PG}(v-1,\mathbb{F}_q)$ and X a $(v-j)$-subspace of $\mathbb{F}_q^v$ with $1 \leq j < r$. Then the restriction $\mathscr{C}\cap X$ is q^{r-j}-divisible.*

Proof. By induction, it suffices to consider the case $j = 1$, i.e., $X = H$ is a hyperplane in $\mathrm{PG}(v-1,\mathbb{F}_q)$.

The hyperplanes of $\mathrm{PG}(H)$ are the $(v-2)$-subspaces of $\mathbb{F}_q^v$ contained in H. Hence the assertion is equivalent to $\#\mathscr{C}\cap U \equiv \#\mathscr{C} = u \pmod{q^{r-1}}$ for every $(v-2)$-subspace $U \subset \mathbb{F}_q^v$. By assumption we have $\#(\mathscr{C}\cap H_i) \equiv u \pmod{q^r}$ for the $q+1$ hyperplanes $H_1,\dots,H_{q+1}$ lying above U. This gives

$$(q+1)u \equiv \sum_{i=1}^{q+1} \#(\mathscr{C}\cap H_i) = q\cdot\#(\mathscr{C}\cap U) + \#\mathscr{C} \equiv q\cdot\#(\mathscr{C}\cap U) + u \pmod{q^r}$$

and hence $u \equiv \#(\mathscr{C}\cap U) \pmod{q^{r-1}}$, as claimed. □ □

As mentioned at the beginning of this section, the set of holes of a partial spread provides an example of a q^r-divisible set. The precise statement is given in the following theorem, which is formulated for general vector space partitions.

Theorem 8. *(i) Let $\mathscr{C}$ be a vector space partition of $\mathbb{F}_q^v$ of type $t^{m_t}\cdots s^{m_s}1^{m_1}$ with $m_s > 0$ (i.e., $\mathscr{C}$ has a member of dimension > 1 and s chosen as the smallest such dimension). Then the points (i.e., 1-subspaces) in $\mathscr{C}$ form a q^{s-1}-divisible set.*
(ii) The holes of a partial k-spread in $\mathbb{F}_q^v$ form a q^{k-1}-divisible set.

Proof. It is immediate that (i) implies (ii). For the proof of (i) let H be a hyperplane of $\mathrm{PG}(v-1,\mathbb{F}_q)$. The points in $\mathscr{P}\setminus H$ are partitioned into the affine subspaces $X\setminus H$ for those $X\in\mathscr{C}$ satisfying $X \nsubseteq H$. If such a t-subspace X is not a point, we have $t \geq s$ and hence $\#(X\setminus H) = q^{t-1} \equiv 0 \pmod{q^{s-1}}$. Moreover, we also have $\#(\mathscr{P}\setminus H) = q^{v-1} \equiv 0 \pmod{q^{s-1}}$. It follows that the number of points in $\mathscr{C}$ that are not contained in H is divisible by q^{s-1} as well, completing the proof. □ □

Theorem 8 explains our motivation for studying q^r-divisible points sets in $\mathrm{PG}(v-1,\mathbb{F}_q)$. Before delving deeper into this topic, we pause for a few example applications to partial spreads, which may help advertising our approach.

First we consider the problem of improving the upper bound (1) for the size of a partial k-spread in $\mathrm{PG}(v-1,\mathbb{F}_q)$. The bound is equivalent to $\#\mathscr{C} \geq \frac{q^r-1}{q-1}$ for the corresponding hole sets, which are q^{k-1}-divisible by Theorem 8(ii). But the smallest nontrivial q^{k-1}-divisible point sets in $\mathrm{PG}(v-1,\mathbb{F}_q)$ are the k-subspaces of $\mathbb{F}_q^v$, since these are associated to

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[28]Readers may have noticed that, curiously, the 3rd standard equation (which characterizes projective codes) was not used at all in the proof.

[29]By this we mean that Δ is the largest divisor in the sense of Definition 1 or Theorem 7.

[30]If $t = \lfloor r \rfloor$ and $r' \in \{0,1,\dots,e-1\}$ is defined by $r = t + r'/e$, then the union of $p^{r'}$ parallel affine subspaces of dimension $t+1$ has this property.

the constant-weight-q^{k-1} simplex code.[31] Thus $\#\mathscr{C} \geq \frac{q^k-1}{q-1} > \frac{q^r-1}{q-1}$, and equality in (1) is not possible. Together with Theorem 2 this already gives the numbers $\mathrm{A}_2(tk+1,2k;k)$.

The preceding argument gives $\mathrm{A}_2(8,6;3) \leq 35$, and as a second application, we now exclude the existence of a partial plane spread of size 35 in $\mathbb{F}_2^8$. As already mentioned, this also follows from the Drake-Freeman bound (Theorem 4) and forms an important ingredient in the determination of the numbers $\mathrm{A}_2(v,6;3)$. The hole set $\mathscr{C}$ of such a partial plane spread has size $2^8-1-35\cdot 7=10$ and is 4-divisible, i.e., it meets every hyperplane in 2 or 6 points.

We claim that $\dim\langle\mathscr{C}\rangle = 4$. The inequality $\dim\langle\mathscr{C}\rangle \geq 4$ is immediate from $\#\mathscr{C} = 10$. The reverse inequality follows from the fact that the linear code C associated with $\mathscr{C}$ is doubly-even, hence self-orthogonal, but cannot be self-dual.[32]

Given that $\dim\langle\mathscr{C}\rangle = 4$, the existence of $\mathscr{C}$ is readily excluded using the standard equations:

$$\begin{aligned} a_2 &+ a_6 &= 15,\\ 2a_2 &+ 6a_6 &= 10\cdot 7,\\ \tbinom{2}{2}a_2 &+ \tbinom{6}{2}a_6 &= \tbinom{10}{2}\cdot 3. \end{aligned} \tag{10}$$

The unique solution of the first two equations is $a_2 = 5$, $a_6 = 10$ (corresponding to the 2-fold repetition of the $[5,4,2]$ even-weight code), but it does not satisfy the third equation (since this code is not projective).[33]

As already mentioned, the standard equations in Lemma 6 have a natural generalization in the language of linear codes. To this end let $\mathscr{C}$ be a point set of size $\#\mathscr{C} = n$ in $\mathrm{PG}(k-1,\mathbb{F}_q)$, which is spanning[34], and C the corresponding projective linear $[n,k]$ code over $\mathbb{F}_q$. The hyperplane spectrum $(a_i)_{0\leq i\leq n}$ of $\mathscr{C}$ and the weight distribution $(A_i)_{0\leq i\leq n}$ of C are related by $A_i = (q-1)a_{n-i}$ for $1 \leq i \leq n$ (supplemented by $A_0 = 1$, $a_n = 0$) and hence provide the same information about $\mathscr{C}$. The famous *MacWilliams Identities*, [51]

$$\sum_{j=0}^{n-i}\binom{n-j}{i}A_j = q^{k-i}\cdot\sum_{j=0}^{i}\binom{n-j}{n-i}A_j^\perp \quad \text{for } 0 \leq i \leq n, \tag{11}$$

relate the weight distributions (A_i), $(A_i^\perp)$ of the (primal) code C and the dual code $C^\perp = \{\mathbf{y} \in \mathbb{F}_q^n; x_1y_1+\cdots+x_ny_n = 0 \text{ for all } \mathbf{x} \in C\}$. They can be solved for A_i (or $A_i^\perp$), resulting in linear relations whose coefficients are values of *Krawtchouk polynomials*; see, e.g., [38, Ch. 7.2] for details. In our case we have $A_1^\perp = A_2^\perp = 0$, since $C^\perp$ has minimum distance $d^\perp \geq 3$, and the first three equations in (11) are equivalent to the equations in Lemma 6.

Of course the A_i and the $A_i^\perp$ in (11) have to be non-negative integers. Omitting the integrality condition yields the so-called *linear programming method*, see e.g. [38, Section 2.6], where the A_i and $A_i^\perp$ are variables satisfying the mentioned constraints.[35] Given some further constraints on the weights of the code and/or the dual code, one may check whether the corresponding polyhedron contains non-negative rational solutions. In general, this is a very powerful approach and was used to compute bounds for codes with a given minimum distance; see [14, 50]. Here we consider a subset of the MacWilliams identities and use analytical arguments.[36]

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31 This follows, e.g., by applying the Griesmer bound to the associated linear code, which has minimum distance $\geq q^{k-1}$ and dimension $\geq k$.

32 Just recall that the length of any doubly-even self-dual binary code must be a multiple of 8.

33 Adding $20 = 5\cdot 4$, which accounts for the 5 pairs of equal points in the code, to the right-hand side "corrects" the third equation.

34 This assumption is necessary for the relation $A_i = (q-1)a_{n-i}$ to hold.

35 Typically, the $A_i^\perp$ are removed from the formulation using the explicit formulas based on the Krawtchouk polynomials, which may of course also be done automatically in the preprocessing step of a customary linear programming solver.

36 The use of a special polynomial, like we will do, is well known in the context of the linear programming method, see e.g. [8, Section 18.1].

By considering the average number of points per hyperplane, we can guarantee the existence of a hyperplane containing a relatively small number of points of $\mathcal{C}$. If this number is nonzero, Lemma 7 allows us to lower-bound $\#\mathcal{C}$ by induction.[37]

Lemma 8. *Suppose that $\mathcal{C} \neq \emptyset$ in $\mathrm{PG}(v-1,\mathbb{F}_q)$ is q^r-divisible with $\#\mathcal{C} = a\cdot q^{r+1}+b$ for some $a,b\in\mathbb{Z}$ and $y\in\mathbb{N}_0$ with $y\equiv(q-1)b \pmod{q^{r+1}}$. Then there exists a hyperplane H such that $\#(\mathcal{C}\cap H)\le(a-1)\cdot q^r+\frac{b+y}{q}\in\mathbb{Z}$ and $\#(\mathcal{C}\cap H)\equiv b \pmod{q^r}$.*

Proof. Set $n=\#\mathcal{C}$ and choose a hyperplane H such that $n' := \#(\mathcal{C}\cap H)$ is minimal. Then, by considering the average number of points per hyperplane, we have

$$n' \le \frac{1}{\begin{bmatrix} v \\ 1\end{bmatrix}_q}\cdot\sum_{\text{hyperplane } H'}\#(\mathcal{C}\cap H') = n\cdot\begin{bmatrix} v-1 \\ 1\end{bmatrix}_q\Big/\begin{bmatrix} v \\ 1\end{bmatrix}_q < \frac{n}{q} = a\cdot q^r+\frac{b}{q}\le a\cdot q^r+\frac{b+y}{q}.$$

Since $n'\equiv b\equiv\frac{b+y}{q} \pmod{q^r}$, this implies $n'\le(a-1)q^r+\frac{b+y}{q}$. □ □

Note that the stated upper bound does not depend on the specific choice of a and b, i.e., there is no need to take a non-negative or *small* b. Choosing y as small as possible clearly gives the sharpest bound.[38] If $b\ge 0$, which one can always achieve by suitably decreasing a, it is always possible to choose $y=(q-1)b$. However, for $q=3$, $r=2$, and $\#\mathcal{C}=1\cdot 3^3+16=43$, i.e., $a=1$ and $b=16$, Lemma 8 with $y=(2-1)16=16$ provides the existence of a hyperplane H with $n'=\#(\mathcal{C}\cap H)\le 0\cdot 3^2+16=16$. Using $y=7$ gives $n'\le 7$ and $n'\equiv 7 \pmod{3^2}$, so that $n'=7$. Applying the argument again yields a subspace of co-dimension 2 containing exactly one hole. Indeed, Equation (9) is needed additionally in order to exclude the possibility of $n'=7$, so that $\mathrm{A}_3(8,6;3)\le 248$, i.e., $\sigma\ge 4$, cf. Theorem 4 stating the same bound.

Corollary 4. *Suppose that $\mathcal{C}\neq\emptyset$ in $\mathrm{PG}(v-1,\mathbb{F}_q)$ is q^r-divisible with $\#\mathcal{C}=a\cdot q^{r+1}+b$ for some $a,b,y\in\mathbb{Z}$ with $y\equiv(q-1)b \pmod{q^{r+1}}$ and $y\ge 1$. Further, let $g\in\mathbb{Q}$ is the largest number with $q^g\mid y$, and $j\in\mathbb{Z}$ satisfies $1\le j<r+1-\max\{0,g-1\}$. Then there exists a $(v-j)$-subspace U such that $\#(\mathcal{C}\cap U)\le(a-j)\cdot q^{r+1-j}+\frac{b+\begin{bmatrix} j \\ 1\end{bmatrix}_q\cdot y}{q^j}$ and $\#(\mathcal{C}\cap U)\equiv b \pmod{q^{r+1-j}}$.*

Proof. In order to apply induction on j, using Lemma 7 and Lemma 8, we need to ensure $n'>0$ in all but the last step. The latter holds due to $pq^g\nmid b$.[39] □ □

Choosing the same value of y in every step, in general is not the optimal way to iteratively apply Lemma 8, even if y is chosen optimal for the first step. To this end, consider a 3^3-divisible set $\mathcal{C}\in\mathrm{PG}(v-1,\mathbb{F}_3)$ with $\#\mathcal{C}=31\cdot 3^4+49=2560$, which indeed exists as the disjoint union of 64 solids is an example. Here $y=17$ with $y\equiv(3-1)\cdot 49 \pmod{3^4}$ is the optimal choice in Lemma 8, so that Corollary 4 guarantees the existence of a subspace U with co-dimension 3, $\#(\mathcal{C}\cap U)\le(31-3)\cdot 3^{4-3}+\frac{49+\begin{bmatrix} 3 \\ 1\end{bmatrix}_3\cdot 17}{3^3}=94$ and $\#(\mathcal{C}\cap U)\equiv 49\equiv 1 \pmod{3^1}$. However, applying Corollary 4 with $j=2$ and $y=17$ guarantees the existence of a subspace U' with co-dimension 2, $\#(\mathcal{C}\cap U')\le(31-2)\cdot 3^{4-2}+\frac{49+\begin{bmatrix} 2 \\ 1\end{bmatrix}_3\cdot 17}{3^2}=29\cdot 3^2+13=30\cdot 3^2+4=274$, and $\#(\mathcal{C}\cap U')\equiv 49\equiv 4 \pmod 9$. Since $\mathcal{C}\cap U'$ is 3^1-divisible and

[37] This result is not *new* at all. In [6] Beutelspacher used such an average argument in his upper bound on the size of partial spreads. Recently, Năstase and Sissokho used it in [57, Lemma 9]. In coding theory it is well known in the context of the Griesmer bound. One may also interpret it as an easy implication of the first two MacWilliams identities, see Lemma 20 and Corollary 9.

[38] Another parameterization for y is given by $y=qb'-b$, where $b'\in\mathbb{Z}$ with $b'\ge\frac{b}{q}$ and $b'\equiv b \pmod{q^{r+1}}$, so that $y\in\mathbb{N}_0$. Due to $b'=\frac{b+y}{q}$, y is minimal if and only if b' is minimal.

[39] The proof shows that the second assertion of the Corollary is true for all $(v-j)$-subspaces U.

$8 \equiv 2 \cdot 4 \pmod{3^2}$, we can apply Lemma 8 with $y = 8$ and deduce the existence of a hyperplane H of U' with $\#(\mathscr{C} \cap (U' \cap H)) \le 29 \cdot 3^1 + \frac{4+8}{3} = 91$ and $\#(\mathscr{C} \cap (U' \cap H)) \equiv 4 \equiv 1 \pmod{3^1}$, while $U' \cap H$ has co-dimension 3.

In the context of partial spreads or, more generally, vector space partitions another parameterization using the number of non-hole elements of the vector space partition turns out to be very useful in order to state a suitable formula for y. In what follows we will say that a vector space partition $\mathscr{P}$ of $\mathbb{F}_q^v$ has *hole-type* (t,s,m_1) if $\mathscr{P}$ has m_1 holes (1-subspaces), $2 \le s \le t < v$, and $s \le \dim(X) \le t$ for all non-holes in $\mathscr{P}$. Additionally, we assume that there is at least one non-hole.

Corollary 5. *Let $\mathscr{P}$ be a vector space partition of $\mathbb{F}_q^v$ of hole-type (t,s,m_1), $l,x \in \mathbb{N}_0$ with $\sum_{i=s}^t m_i = lq^s + x$, and $b,c \in \mathbb{Z}$ with $m_1 = bq^s + c \ge 1$. If $x \ge 2$ and g is the largest integer such that q^g divides $x-1$, then for each $0 \le j \le s - \max\{1,g\}$ there exists a $(v-j)$-dimensional subspace U containing $\widehat{m}_1$ holes with $\widehat{m}_1 \equiv \widehat{c} \pmod{q^{s-j}}$ and $\widehat{m}_1 \le (b-j) \cdot q^{s-j} + \widehat{c}$, where $\widehat{c} = \frac{c + \left[{j \atop 1}\right]_q \cdot (x-1)}{q^j} \in \mathbb{Z}$.*

Proof. We have $\left[{v \atop 1}\right]_q = m_1 + \sum_{i=s}^t m_i \left[{i \atop 1}\right]_q$. Multiplication by $q-1$ and reduction modulo q^s yields $-1 \equiv (q-1)c - x \pmod{q^s}$, allowing us to apply Corollary 4 with $x = y-1$. Observe that the parameters g from Corollary 4 and Corollary 5 differ by at most $1 - 1/e$ if $q = p^e$. □ □

So far, we can guarantee that some subspace contains not *too many* holes, since the average number of holes per subspace would be too large otherwise. The modulo-constraints captured in the definition of a q^r-divisible set enable iterative rounding, thereby sharpening the bounds. First we consider the special case of partial spreads, and then we will derive some non-existence results for vector space partitions with *few* holes.

Lemma 9. *Let $\mathscr{P}$ be a vector space partition of type $k^{m_k}1^{m_1}$ of $\mathbb{F}_q^v$ with $m_k = lq^k + x$, where $l = \frac{q^{v-k} - q^r}{q^k - 1}$, $x \ge 2$, $k = \left[{r \atop 1}\right]_q + 1 - z + u > r$, $q^g \mid x-1$, $q^{g+1} \nmid x-1$, and $g,u,z,r,x \in \mathbb{N}_0$. For $\max\{1,g\} \le y \le k$ there exists a $(v-k+y)$-subspace U with $L \le (z+y-1-u)q^y + w$ holes, where $w = -(x-1)\left[{y \atop 1}\right]_q$ and $L \equiv w \pmod{q^y}$.*

Proof. Due to $m_1 = \left[{v \atop 1}\right]_q - m_k \cdot \left[{k \atop 1}\right]_q = \left[{r \atop 1}\right]_q q^k - \left[{k \atop 1}\right]_q (x-1)$, we have $m_1 = bq^k + c$ for $b = \left[{r \atop 1}\right]_q$ and $c = -\left[{k \atop 1}\right]_q (x-1)$, where $q^{g'} \mid x-1$ if and only if $q^{g'} \mid c$. Setting $s = t = k$ and $j = k - y$, we observe $0 \le j \le k - \max\{1,g\}$, since $\max\{1,g\} \le y \le k$. With this, we apply Corollary 5 and obtain an $(n-k+y)$-subspace U with

$$\begin{aligned} L &\le (b-j) \cdot q^{k-j} + \frac{c + \left[{j \atop 1}\right]_q \cdot (x-1)}{q^j} = (z+y-1-u) \cdot q^y - (x-1) \cdot \frac{\left[{k \atop 1}\right]_q - \left[{k-y \atop 1}\right]_q}{q^{k-y}} \\ &= (z+y-1-u)q^y - (x-1)\begin{bmatrix} y \\ 1 \end{bmatrix}_q = (z+y-1-u)q^y + w \end{aligned}$$

holes, so that $L \le (z+y-1-u)q^y + w$ and $L \equiv w \pmod{q^y}$. □ □

The parameter l is chosen in such a way that $m_k = lq^k + x$ matches the cardinality of the partial k-spread given by the construction in Theorem 2 for $x = 1$. Thus the assumption $x \ge 2$ is no real restriction. Actually, the chosen parameterization using x in Corollary 5 makes it very transparent why the construction of Theorem 2 is *asymptotically optimal*—as stated in Theorem 6. If the dimension k of the elements of the partial spread is large enough, a sufficient number of rounding steps can be performed while the rounding process is stopped at $x = 1$ for the other direction. For *small* k we will not reach the lower bound of the construction of Theorem 2, so that there remains some room for better constructions.

Lemma 10. *Let $\Delta = q^{s-1}$, $m \in \mathbb{Z}$, and $\mathscr{P}$ be a vector space partition of $\mathbb{F}_q^v$ of hole-type (t,s,c). Then, $\tau_q(c,\Delta,m) \cdot \frac{q^{v-2}}{\Delta^2} - m(m-1) \ge 0$ and $\tau_q(c,\Delta,m) \ge 0$, where $\tau_q(c,\Delta,m) =$*

$m(m-1)\Delta^2q^2 - c(2m-1)(q-1)\Delta q + c(q-1)\Big(c(q-1)+1\Big)$. *If* $c>0$, *then* $\tau_q(c,\Delta,m)=0$ *if and only if* $m=1$ *and* $c=\begin{bmatrix}s\\1\end{bmatrix}_q$.

Proof. Adding $(c-m\Delta)\Big(c-(m-1)\Delta\Big)$ times the first, $-\Big(2c-(2m-1)\Delta-1\Big)$ times the second and twice the third standard equation from Lemma 6, and dividing the result by $\Delta^2/(q-1)$ gives $(q-1)\cdot\sum_{h=0}^{\lfloor c/\Delta\rfloor}(m-h)(m-h-1)a_{c-h\Delta} = \tau_q(c,\Delta,m)\cdot\frac{q^{n-2}}{\Delta^2} - m(m-1)$, due to the q^{s-1}-divisibility. We observe $a_i\ge 0$ and $(m-h)(m-h-1)\ge 0$ for all $m,h\in\mathbb{Z}$. If $m\notin\{0,1\}$, then $\tau_q(c,\Delta,m)>0$. Solving $\tau_q(c,\Delta,0)=0$ yields $c\in\left\{0,-\frac{q^s+1}{q-1}\right\}$ and solving $\tau_q(c,\Delta,1)=0$ yields $c\in\left\{0,\begin{bmatrix}s\\1\end{bmatrix}_q\right\}$. □ □

We remark that, in the case of $\tau_q(c,\Delta,m)=0$, their are either no holes at all or the holes form an s-subspace. [10, Theorem 1.B] is quite similar to Lemma 10 and its implications. The multipliers used in the proof can be directly read off from the inverse matrix of

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2-a & b^2-b & c^2-c \end{pmatrix},$$

which is given by

$$\mathbf{A}^{-1} = \frac{1}{(c-a)(c-b)(b-a)}\begin{pmatrix} bc(c-b) & -(c+b-1)(c-b) & (c-b) \\ -ac(c-a) & (c+a-1)(c-a) & -(c-a) \\ ab(b-a) & -(b+a-1)(b-a) & (b-a) \end{pmatrix}$$

for distinct numbers a,b,c. With this, Lemma 10 can be derived in a conceptual way. Consider the linear programming method with just the first three MacWilliams identities. For parameters excluded by Lemma 10 this small linear program is infeasible, which can be seen at a certain basis solution, i.e., a choice of linear inequalities that are satisfied with equality. Solving for these equations, i.e., a change of basis, corresponds to a non-negative linear combination of the inequality system.[40] In the parametric case we have to choose the basis solution also depending on the parameters. Actually, we have implemented a degree of freedom in Lemma 10 using the parameter m. Here, the basis consists of two neighboring non-zero a_i-entries, parameterized by m, and an arbitrary a_i, which plays no role when the resulting equation is solved for all remaining a_i-terms. In this way we end up with an equation of the form $\sum_{h=0}^{\lfloor c/\Delta\rfloor}(m-h)(m-h-1)a_{c-h\Delta}=\beta$, where the a_i and their coefficients are non-negative. The use of the underlying quadratic polynomial is well known and frequently applied in the literature; see the remarks after Theorem 4.

Lemma 11. *For integers* $v>k\ge s\ge 2$ *and* $1\le i\le s-1$, *there exists no vector space partition* $\mathscr{P}$ *of* $\mathbb{F}_q^v$ *of hole-type* (k,s,c), *where* $c=i\cdot q^s-\begin{bmatrix}s\\1\end{bmatrix}_q+s-1$.[41]

Proof. Since we have $c<0$ for $i\le 0$, we can assume $i\ge 1$ in the following. Let, to the contrary, $\mathscr{P}$ be such a vector space partition and apply Lemma 10 with $m=i(q-1)$ onto $\mathscr{P}$. We compute $\tau_q(c,q^{s-1},m)=(m-1-a)q^s+a(a+1)$ using $c(q-1)=q^s(m-1)+a$, where $a:=1+(s-1)(q-1)$. Setting $i=s-1-y$, we have $0\le y\le s-2$ and $\tau_q(c,q^{s-1},m)=-q^s(y(q-1)+2)+(s-1)^2q^2-q(s-1)(2s-5)+(s-2)(s-3)$. If $q=2$, then $y\ge 0$ and $s\ge 2$ yields

[40] If we relax ≥ 0-inequalities by adding some auxiliary variable on the left hand side and the minimization of this variable, we can remove the infeasibility, so that we apply the duality theorem of linear programming. Then, the mentioned multipliers for the inequalities are given as the solution values of the dual problem.

[41] For more general non-existence results of vector space partitions see e.g. [28, Theorem 1] and the related literature. Actually, we do not need the assumption of an underlying vector space partition of the mentioned type. The result is generally true for q^{s-1}-divisible codes, since the parameter x is just a nice technical short-cut to ease the notation.

$$\tau_2(c,2^{s-1},m) = -2^s(y+2)+s^2+s \le \left(s^2-s-2^s\right)+(2s-2^s) < 0.$$

If $s=2$, then we have $y=0$ and $\tau_q(c,q^{s-1},m) = -q^2+q < 0$. If $q,s \ge 3$, then we have $q(2s-5) \ge s-3$, so that $\tau_q(c,q^{s-1},m) \le -2q^s+(s-1)^2q^2 \le -2\cdot 3^{s-2}q^2+(s-1)^2q^2$ due to $y \ge 0$ and $q \ge 3$. Since $2\cdot 3^{s-2} > (s-1)^2$ for $s \ge 3$, we have $\tau_q(c,q^{s-1},m) < 0$ in all cases. Thus, Lemma 10 yields a contradiction, since $q^{n-2s} > 0$ and $m(m-1) \ge 0$ for every integer m. □ □

Now we are ready to present the first improved (compared to Theorem 4) upper bound for partial spreads, which also covers Theorem 6 setting $z=0$.

Theorem 9. *For integers $r \ge 1$, $t \ge 2$, $u \ge 0$, and $0 \le z \le \left[{r \atop 1}\right]_q/2$ with $k = \left[{r \atop 1}\right]_q + 1 - z + u > r$ we have $\mathrm{A}_q(v,2k;k) \le lq^k + 1 + z(q-1)$, where $l = \frac{q^{v-k}-q^r}{q^k-1}$ and $v = kt+r$.*

Proof. Apply Lemma 9 with $x = 2+z(q-1) \ge 2$ in order to deduce the existence of a $(v-k+y)$-subspace U with $L \le (z+y-1-u)q^y - (x-1)\left[{y \atop 1}\right]_q$ holes, where $L \equiv -(x-1)\left[{y \atop 1}\right]_q \pmod{q^y}$. Now, we set $y = z+1$. Observe that $q^g \mid x-1$ implies $g \le z < y$ and we additionally have $1 \le y = z+1 \le \left[{r \atop 1}\right]_q + 1 - z \le t$. If $z=0$, then $y=1$, $x=2$, and $L \le -uq-1 < 0$. For $z \ge 1$, we apply Lemma 11 to the subspace U with $s=y$, $c = (z+y-1-u)q^y - (x-1)\left[{y \atop 1}\right]_q - jq^y = (y-1-j-u)q^y - \left[{y \atop 1}\right]_q + y - 1$ for some $j \in \mathbb{N}_0$, and $i = y-1-j-u \in \mathbb{Z}$. Thus, $\mathrm{A}_q(n,2k;k) \le lq^k + x - 1$. □ □

The case $z=0$ covers Theorem 6. The non-negativity of the number of holes in a certain carefully chosen subspace is sufficient to prove this fact. The case $z=1$ was announced in [57, Lemma 10] and proven in [56]. Since the known constructions for partial k-spreads give $\mathrm{A}_q(kt+r,2k;k) \ge lq^k+1$, see e.g. [6] or Theorem 2, Theorem 9 is tight for $k \ge \left[{r \atop 1}\right]_q + 1$ and $\mathrm{A}_2(8,6;3) = 34$.

So far Lemma 10 was just applied in the case of Lemma 11 excluding the existence of some very special vector space partitions. Next, we look at a subspace and consider the number of holes, i.e., we apply Lemma 9 giving us the freedom to choose the dimension of the subspace. Then Lemma 10, stating that a certain quadratic polynomial is non-negative, can be applied. By minimizing this function in terms of the free parameter m, we obtain the following result.

Theorem 10. *For integers $r \ge 1$, $t \ge 2$, $y \ge \max\{r,2\}$, $z \ge 0$ with $\lambda = q^y$, $y \le k$, $k = \left[{r \atop 1}\right]_q + 1 - z > r$, $v = kt+r$, and $l = \frac{q^{v-k}-q^r}{q^k-1}$, we have*

$$\mathrm{A}_q(v,2k;k) \le lq^k + \left\lceil \lambda - \frac{1}{2} - \frac{1}{2}\sqrt{1+4\lambda\left(\lambda-(z+y-1)(q-1)-1\right)} \right\rceil.$$

Proof. From Lemma 9 we conclude $L \le (z+y-1)q^y - (x-1)\left[{y \atop 1}\right]_q$ and $L \equiv -(x-1)\left[{y \atop 1}\right]_q \pmod{q^y}$ for the number of holes of a certain $(v-k+y)$-subspace U. Using the notation of Lemma 9, $\mathscr{P} \cap U := \{P \cap U \mid P \in \mathscr{P}\}$ is of hole-type (k,y,L) if $y \ge 2$. Next, we will show that $\tau_q(c,\Delta,m) \le 0$, where $\Delta = q^{y-1}$ and $c = iq^y - (x-1)\left[{y \atop 1}\right]_q$ with $1 \le i \le z+y-1$, for suitable integers x and m. Note that, in order to apply Lemma 9, we have to satisfy $x \ge 2$ and $y \ge g$ for all integers g with $q^g \mid x-1$. Applying Lemma 10 then gives the desired contradiction, so that $\mathrm{A}_q(n,2k;k) \le lq^k + x - 1$.

We choose[42] $m = i(q-1)-(x-1)+1$, so that $\tau_q(c,\Delta,m) = x^2-(2\lambda+1)x+\lambda(i(q-1)+2)$. Solving $\tau_q(c,\Delta,m)=0$ for x gives $x_0 = \lambda + \frac{1}{2} \pm \frac{1}{2}\theta(i)$, where

$$\theta(i) = \sqrt{1-4i\lambda(q-1)+4\lambda(\lambda-1)}.$$

[42] Solving $\frac{\partial \tau_q(c,\Delta,m)}{\partial m} = 0$, i.e., minimizing $\tau_q(c,\Delta,m)$, yields $m = i(q-1)-(x-1)+\frac{1}{2}+\frac{x-1}{q^y}$. For $y \ge r$ we can assume $x-1 < q^y$ due to Theorem 2, so that up-rounding yields the optimum integer choice. For $y < r$ the interval $\left[\lambda+\frac{1}{2}-\frac{1}{2}\theta(i), \lambda+\frac{1}{2}+\frac{1}{2}\theta(i)\right]$ may contain no integer.

We have $\tau_q(c,\Delta,m) \le 0$ for $|2x-2\lambda-1| \le \theta(i)$. We need to find an integer $x \ge 2$ such that this inequality is satisfied for all $1 \le i \le z+y-1$. The strongest restriction is attained for $i = z+y-1$. Since $z+y-1 \le \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ and $\lambda = q^y \ge q^r$, we have $\theta(i) \ge \theta(z+y-1) \ge 1$, so that $\tau_q(c,\Delta,m) \le 0$ for $x = \left\lceil \lambda + \frac{1}{2} - \frac{1}{2}\theta(z+y-1) \right\rceil$. (Observe $x \le \lambda + \frac{1}{2} + \frac{1}{2}\theta(z+y-1)$ due to $\theta(z+y-1) \ge 1$.) Since $x \le \lambda + 1$, we have $x - 1 \le \lambda = q^y$, so that $q^g \mid x-1$ implies $g \le y$ provided $x \ge 2$. The latter is true due to $\theta(z+y-1) \le \sqrt{1-4\lambda(q-1)+4\lambda(\lambda-1)} \le \sqrt{1+4\lambda(\lambda-2)} < 2(\lambda-1)$, which implies $x \ge \left\lceil \frac{3}{2} \right\rceil = 2$.

So far we have constructed a suitable $m \in \mathbb{Z}$ such that $\tau_q(c,\Delta,m) \le 0$ for

$$x = \left\lceil \lambda + \frac{1}{2} - \frac{1}{2}\theta(z+y-1) \right\rceil .$$

If $\tau_q(c,\Delta,m) < 0$, then Lemma 10 gives a contradiction, so that we assume $\tau_q(c,\Delta,m) = 0$ in the following. If $i < z+y-1$ we have $\tau_q(c,\Delta,m) < 0$ due to $\theta(i) > \theta(z+y-1)$, so that we assume $i = z+y-1$. Thus, $\theta(z+y-1) \in \mathbb{N}_0$. However, we can write $\theta(z+y-1)^2 = 1+4\lambda\left(\lambda-(z+y-1)(q-1)-1\right) = (2w-1)^2 = 1+4w(w-1)$ for some integer w. If $w \notin \{0,1\}$, then $\gcd(w,w-1) = 1$, so that either $\lambda = q^y \mid w$ or $\lambda = q^y \mid w-1$. Thus, in any case, $w \ge q^y$, which is impossible since $(z+y-1)(q-1) \ge 1$. Finally, $w \in \{0,1\}$ implies $w(w-1) = 0$, so that $\lambda-(z+y-1)(q-1)-1 = 0$. Thus, $z+y-1 = \begin{bmatrix} y \\ 1 \end{bmatrix}_q \ge \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ since $y \ge r$. The assumptions $y \le k$ and $k = \begin{bmatrix} r \\ 1 \end{bmatrix}_q + 1 - z$ imply $z+y-1 = \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ and $y = r$. This gives $k = r$, which is excluded. □ □

An example where Theorem 10 is strictly superior to the results of [56, Theorem 6,7] is given by $\mathrm{A}_3(15,12;6) \le 19695$.[43] Setting $y = k$, we obtain Theorem 4. Compared to [10, 19], the new ingredients essentially are q^r-divisible sets and Corollary 5, which allows us to choose $y < k$. Theorem 4, e.g., gives $\mathrm{A}_2(15,12;6) \le 516$, $\mathrm{A}_2(17,14;7) \le 1028$, and $\mathrm{A}_9(18,16;8) \le 3486784442$, while Theorem 10 gives $\mathrm{A}_2(15,12;6) \le 515$, $\mathrm{A}_2(17,14;7) \le 1026$, and $\mathrm{A}_9(18,16;8) \le 3486784420$. For $2 \le q \le 9$, $5 \le k \le 19$, there are 66 improvements in total, i.e., almost 19%, and the maximum gap is 22. Next, we provide an estimation of the bound of Theorem 4.

Lemma 12. *For integers $1 \le r < k$ and $q \ge 2$ we have*

$$2q^k - q^r - \frac{q^{2r-k}}{\underline{b}} < \sqrt{1+4q^k(q^k-q^r)} \le 2q^k - q^r - \frac{q^{2r-k}}{\overline{b}},$$

where $\underline{b} = \frac{3+2\sqrt{2}}{2} > 2.91$ and $\overline{b} = \frac{16}{3} < 5.34$.

Proof. Due to $q \ge 2$ and $k \ge r+1 \ge 2$, we have

$$1+4q^k(q^k-q^r) > 4q^k(q^k-q^r) - q^{2r} \cdot \overbrace{\left(\frac{4}{\underline{b}} - 1 - \frac{2}{\underline{b}q} - \frac{1}{\underline{b}^2 q^2}\right)}^{\ge 0}$$

$$\ge 4q^k(q^k-q^r) - \frac{4}{\underline{b}}q^{2r} + q^{2r} + \frac{2}{\underline{b}}q^{3r-k} + \frac{1}{\underline{b}^2}q^{4r-2k} = \left(2q^k - q^r - \frac{q^{2r-k}}{\underline{b}}\right)^2 .$$

Similarly, $1+4q^k(q^k-q^r) \le 4q^k(q^k-q^r) - q^{2r} \cdot \left(\frac{4}{\overline{b}} - 1\right) \le \left(2q^k - q^r - \frac{q^{2r-k}}{\overline{b}}\right)^2$. □ □

Corollary 6. *For integers $1 \le r < k$ and $t \ge 2$ we have $\mathrm{A}_q(kt+r,2k;k) < lq^k + \frac{q^r}{2} + \frac{1}{2} + \frac{q^{2r-k}}{3+2\sqrt{2}}$, where $l = \frac{q^{(t-1)k+r}-q^r}{q^k-1}$. If $k \ge 2r$, then $\mathrm{A}_q(kt+r,2k;k) < lq^k + 1 + \frac{q^r}{2}$.*

Corollary 7. *For integers $r \ge 1$, $t \ge 2$, and $u,z \ge 0$ with $k = \begin{bmatrix} r \\ 1 \end{bmatrix}_q + 1 - z + u > r$ we have $\mathrm{A}_q(v,2k;k) \le lq^k + 1 + z(q-1)$, where $l = \frac{q^{v-k}-q^r}{q^k-1}$ and $v = kt+r$.*

[43] For $2 \le q \le 9, 1 \le v,k \le 100$ the bounds of [56, Theorem 6,7] are covered by Theorem 10 and Corollary 7. In many cases the bounds coincide.

Proof. Using Corollary 6, we can remove the upper bound $z \le \begin{bmatrix} r \\ 1 \end{bmatrix}_q/2$ from Theorem 9. If $z > \begin{bmatrix} r \\ 1 \end{bmatrix}_q/2$, then $z \ge \begin{bmatrix} r \\ 1 \end{bmatrix}_q/2 + 1/2$, so that $\mathrm{A}_q(v,2k;k) < lq^k + 1 + \frac{q^r}{2} \le lq^k + 1 + \frac{q^r-1}{2} + \frac{q-1}{2} \le lq^k + 1 + z(q-1)$ for $k \ge 2r$. Thus, we can assume $r+1 \le k \le 2r-1$ and $r \ge 2$. With this, we have $z \ge \begin{bmatrix} r \\ 1 \end{bmatrix}_q - 2(r-1)$ and $lq^k + 1 + z(q-1) \ge lq^k + q^r - 2(q-1)(r-1)$. It remains to show $lq^k + q^r - 2(q-1)(r-1) \ge lq^k + \frac{q^r}{2} + \frac{1}{2} + \frac{q^{2r-k}}{3+2\sqrt{2}} \ge lq^k + \frac{q^r}{2} + \frac{1}{2} + \frac{q^{r-1}}{3+2\sqrt{2}}$, i.e., $q^r \ge 1 + \frac{2q^{r-1}}{3+2\sqrt{2}} + 4(q-1)(r-1)$. The latter inequality is valid for all pairs (r,q) except $(2,2)$, $(2,3)$, and $(3,2)$. In those cases it can be verified directly that $lq^k + 1 + z(q-1)$ is not strictly less than the upper bound of Theorem 4. Indeed, both bounds coincide. □ □

We remark that the first part of Corollary 6 can be written as $\sigma \ge \frac{q^r-1}{2} - \frac{q^{2r-k}}{3+2\sqrt{2}}$. Unfortunately, Theorem 10 is not capable to obtain $\sigma \ge \lfloor (q^r-1)/2 \rfloor$. For $\mathrm{A}_2(17,12;6)$, i.e., $q=2$ and $r=5$, it gives $\sigma \ge 13$ while $\lfloor (q^r-1)/2 \rfloor = 15$. In Lemma 23 we give a cubic analog to Lemma 10, which yields $\sigma \ge 14$ for these parameters.

4. CONSTRUCTIONS FOR q^r-DIVISIBLE SETS

First note that we can embed every Δ-divisible point set $\mathscr{C}$ in $\mathrm{PG}(v-1,\mathbb{F}_q)$ into ambient spaces with dimension larger than v and, conversely, replace $\mathbb{F}_q^v$ by the span $\langle \mathscr{C} \rangle$ without destroying the Δ-divisibility. Since in this sense v is not determined by $\mathscr{C}$, we will refer to $\mathscr{C}$ as a *Δ-divisible point set over* $\mathbb{F}_q$. In the sequel we develop a few basic constructions of q^r-divisible sets. For the statement of the first lemma recall our convention that subspaces of $\mathbb{F}_q^v$ are identified with subsets of the point set $\mathscr{P}$ of $\mathrm{PG}(v-1,\mathbb{F}_q)$.

Lemma 13. *Every k-subspace $\mathscr{C}$ of $\mathrm{PG}(v-1,\mathbb{F}_q)$ with $k \ge 2$ is q^{k-1}-divisible.*

Proof. By the preceding remark we may assume $k = v$ and hence $\mathscr{C} = \mathscr{P}$. In this case the result is clear, since $\#\mathscr{P} - \#H = q^{v-1}$ for each hyperplane H. □ □

In fact a k-subspace of $\mathbb{F}_q^v$ is associated to the k-dimensional simplex code over $\mathbb{F}_q$ and Lemma 13 is well-known.

For a point set $\mathscr{C}$ in $\mathrm{PG}(v-1,\mathbb{F}_q)$ we denote by $\chi_{\mathscr{C}}$ its characteristic function, i.e., $\chi_{\mathscr{C}} \colon \mathscr{P} \to \{0,1\} \subset \mathbb{Z}$ with $\chi_{\mathscr{C}}(P) = 1$ if and only if $P \in \mathscr{C}$.

Lemma 14. *Let $\mathscr{C}_i$ be Δ_i-divisible point sets in $\mathrm{PG}(v-1,\mathbb{F}_q)$ and $a_i \in \mathbb{Z}$ for $1 \le i \le m$. If $\mathscr{C} \subseteq \mathscr{P}$ satisfies $\chi_{\mathscr{C}} = \sum_{i=1}^m a_i \chi_{\mathscr{C}_i}$ then $\mathscr{C}$ is $\gcd(a_1\Delta_1,\ldots,a_m\Delta_m)$-divisible.*

Proof. We have $\#\mathscr{C} = \sum_{i=1}^m a_i \cdot \#\mathscr{C}_i$ and $\#(\mathscr{C} \cap H) = \sum_{i=1}^m a_i \cdot \#(\mathscr{C}_i \cap H)$ for each hyperplane H. Since $\#(\mathscr{C}_i \cap H) \equiv \#\mathscr{C}_i \pmod{\Delta_i}$, the result follows. □ □

Lemma 14 shows in particular that the union of mutually disjoint q^r-divisible sets is again q^r-divisible. Another (well-known) corollary is the following, which expresses the divisibility properties of the MacDonald codes.[44]

Corollary 8. *Let $X \subsetneq Y$ be subspaces of $\mathbb{F}_q^v$ and $\mathscr{C} = Y \setminus X$. If $\dim(X) = s$ then $\mathscr{C}$ is q^{s-1}-divisible.*

In particular affine k-subspaces of $\mathbb{F}_q^v$ are q^{k-1} divisible.

Lemma 15. *Let $\mathscr{C}_1 \in \mathscr{P}_1$, $\mathscr{C}_2 \in \mathscr{P}_2$ be q^r-divisible point sets in $\mathrm{PG}(v_1-1,\mathbb{F}_q)$, respectively, $\mathrm{PG}(v_2-1,\mathbb{F}_q)$. Then there exists a q^r-divisible set $\mathscr{C}$ in $\mathrm{PG}(v_1+v_2-1,\mathbb{F}_q)$ with $\#\mathscr{C} = \#\mathscr{C}_1 + \#\mathscr{C}_2$.*

Proof. Embed the point sets $\mathscr{C}_1,\mathscr{C}_2$ in the obvious way into $\mathrm{PG}(\mathbb{F}_q^{v_1} \times \mathbb{F}_q^{v_2}) \cong \mathrm{PG}(v_1+v_2-1,\mathbb{F}_q)$, and take $\mathscr{C}$ as their union. □ □

[44] The generalization to more than one “removed” subspace is also quite obvious and expresses the divisibility properties of optimal linear codes of type BV in the projective case [3, 33, 48].

Let us note that the embedding dimension v in Lemma 15 is usually not the smallest possible, and the isomorphism type of $\mathscr{C}$ is usually not determined by $\mathscr{C}_1$ and $\mathscr{C}_2$.[45]

In analogy to the *Frobenius Coin Problem*, cf. [7, 12, 27], we define $\mathrm{F}(q,r)$ as the smallest positive integer such that a q^r-divisible set over $\mathbb{F}_q$ (i.e., with some ambient space $\mathbb{F}_q^v$) with cardinality n exists for all integers $n > \mathrm{F}(q,r)$. Using Lemma 13, Corollary 8, and Lemma 15, we conclude that $\mathrm{F}(q,r) \le \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q \cdot q^{r+1} - \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q - q^{r+1}$, the largest integer not representable as $a_1 \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q + a_2 q^{r+1}$ with $a_1, a_2 \in \mathbb{Z}_{\ge 0}$.[46] The bound may also be stated as $\mathrm{F}(q,r) \le \sum_{i=r+2}^{2r+1} q^i - \sum_{i=0}^{r} q^i$.

As the disjoint union of q^r-divisible sets is again q^r-divisible, one obtains a wealth of constructions. Consequently, the q^r-divisible point sets not arising in this way are of particular interest. They are called *indecomposable*.

The next construction uses the concept of a "sunflower" of subspaces, which forms the q-analogue of the Δ-systems, or sunflowers, considered in extremal set theory [22].[47]

Definition 2. *Let X be a subspace of $\mathbb{F}_q^v$ and $t \ge 2$ an integer. A t-*sunflower *in $\mathbb{F}_q^v$ with* center *X is a set $\{Y_1, \dots, Y_t\}$ of subspaces of $\mathbb{F}_q^v$ satisfying $Y_i \ne X$ and $Y_i \cap Y_j = X$ for $i \ne j$. The point sets $Y_i \setminus X_i$ are called* petals *of the sunflower.*

Lemma 16. *(i) The union of the petals of a q-sunflower in $\mathbb{F}_q^v$ with r-dimensional center forms a q^r-divisible point set.*
(ii) The union of the petals and the center of a $q+1$-sunflower in $\mathbb{F}_q^v$ with r-dimensional center forms a q^r-divisible point set.

Proof. (i) Let $\mathscr{F} = \{Y_1, \dots, Y_q\}$ and $\mathscr{C} = \bigcup_{i=1}^{q} (Y_i \setminus X_i) = (\bigcup \mathscr{F}) \setminus X$. We have $\chi_{\mathscr{C}} = \sum_{i=1}^{q} \chi_{Y_i} - q\chi_X$. Since $\dim(Y_i) \ge r+1$, Y_i is q^r-divisible, and so is $q\chi_X$. Hence, by Lemma 14, $\mathscr{C}$ is q^r-divisible as well.

(ii) follows from (i) by adding one further space Y_{q+1} to $\mathscr{F}$. □ □

Lemma 17. *Let $r \ge 1$ be an integer and $1 \le i \le q^r + 1$. There exists a q^r-divisible set $\mathscr{C}_i$ over $\mathbb{F}_q$ with $\#\mathscr{C}_i = \begin{bmatrix} 2r \\ 1 \end{bmatrix}_q + i \cdot \left(q^{r+1} - q^r - \begin{bmatrix} r \\ 1 \end{bmatrix}_q \right)$.*

Proof. Let $Y = \mathbb{F}_q^{2r}$ and $X_1, \dots, X_{q^r+1}$ an r-spread in Y. After embedding Y in a space $\mathbb{F}_q^v$ of sufficiently large dimension v, it is possible to choose q-sunflowers $\mathscr{F}_1, \dots, \mathscr{F}_{q^r+1}$ in $\mathbb{F}_q^v$ with the following properties: $Y \in \mathscr{F}_i$ for all i; $\dim(Z) = r+1$ for $Z \in \mathscr{F}_i \setminus \{Y\}$; $\mathscr{F}_i$ has center X_i; petals in different sunflowers $\mathscr{F}_i$ and $\mathscr{F}_j$ are either equal (to Y) or disjoint. Having made such a choice, we set $\mathscr{C}_i = (\bigcup \mathscr{F}_1 \cup \dots \cup \bigcup \mathscr{F}_i) \setminus (X_1 \cup \dots \cup X_i)$ for $1 \le i \le q^r + 1$. Then $\chi_{\mathscr{C}_i} = \sum_{Z \in \mathscr{F}_1 \cup \dots \cup \mathscr{F}_i} \chi_Z - q\chi_{X_1} - \dots - q\chi_{X_i}$ is q^r-divisible (again by Lemma 14), and $\#\mathscr{C}_i$ is as asserted. □ □

Replacing S by some arbitrary q^1-divisible set, we similarly obtain:

Lemma 18. *Let $\mathscr{C}'$ be a q^1-divisible set of cardinality n, then there exist q^1-divisible sets of cardinality $n + i \cdot (q^2 - q - 1)$ for all $0 \le i \le n$.*

Our last construction in this section uses the concept of a cone.

Definition 3. *Let X, Y be complementary subspaces of $\mathbb{F}_q^v$ with $\dim(X) = s$, $\dim(Y) = t$ (hence $v = s + t$) and $\mathscr{B}$ a set of points in $\mathrm{PG}(Y)$. The* cone *with* vertex *X and* base *$\mathscr{B}$ is the point set $\mathscr{C} = \bigcup_{P \in \mathscr{B}} (X + P)$.*

Lemma 19. *Let $\mathscr{B}$ be a q^r-divisible point set in $\mathrm{PG}(v-1, \mathbb{F}_q)$ with $\#\mathscr{B} = m$ and $s \ge 1$ an integer.*

[45] If not both $\mathscr{C}_1$ and $\mathscr{C}_2$ are subspaces, then disjoint embeddings into a geometry $\mathrm{PG}(v-1, \mathbb{F}_q)$ with $v < \dim\langle\mathscr{C}_1\rangle + \dim\langle\mathscr{C}_2\rangle$ exist as well.

[46] Note that $\gcd\left(\begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q, q^{r+1} \right) = 1$ and recall the solution of the ordinary Frobenius Coin Problem.

[47] Our sunflowers need not have constant dimension, however.

(i) If $m \equiv 0 \pmod{q^{r+1}}$ *then there exists a* q^{r+s}*-divisible point set* $\mathscr{C}$ *in* $\mathrm{PG}(v+s-1,\mathbb{F}_q)$ *of cardinality* $\#\mathscr{C} = mq^s$.

(ii) If $m(q-1) \equiv -1 \pmod{q^{r+1}}$ *then there exists a* q^{r+s}*-divisible point set* $\mathscr{C}$ *in* $\mathrm{PG}(v+s-1,\mathbb{F}_q)$ *of cardinality* $\#\mathscr{C} = \begin{bmatrix} s \\ 1 \end{bmatrix}_q + mq^s$.

Proof. Embed $\mathbb{F}_q^v$ into $\mathbb{F}_q^{v+s}$ as Y and consider a cone $\mathscr{K}$ in $\mathrm{PG}(v+s-1,\mathbb{F}_q)$ with base $\mathscr{B}$ and s-dimensional vertex X. The hyperplanes $H \supseteq X$ satisfy $\#(\mathscr{K}\setminus H) = q^s \cdot \#(\mathscr{K}\setminus H) \equiv 0 \pmod{q^{r+s}}$. The hyperplanes $H \not\supseteq X$ intersect $X+P$, $P \in \mathscr{B}$, in an s-subspace $\neq X$, hence contain $\begin{bmatrix} s \\ s-1 \end{bmatrix}_q =$ points in X and q^{s-1} points in $\mathscr{K}\setminus X$. It follows that $\#(\mathscr{K}\setminus H) = q^{s-1} + m(q^s - q^{s-1}) = (1+m(q-1))q^{s-1}$. Thus in Case (ii) we can take $\mathscr{C} = \mathscr{K}$ and in Case (i) we can take $\mathscr{C} = \mathscr{K}\setminus X$.[48] □ □

The preceding constructions can be combined in certain nontrivial ways to yield further constructions of q^r-divisible point sets. We will return to this topic in Section 6.1.

Nonetheless, we are only scratching the surface of a vast subject. Projective two-weight codes with weights w_1, w_2 satisfying $w_2 > w_1 + 1$ are q^r-divisible by Delsarte's Theorem [15, Cor. 2]. This yields many further examples of q^r-divisible point sets; see [13] and the online-table at `http://moodle.tec.hkr.se/\textasciitildechen/research/2-weight-codes`. Codes meeting the Griesmer bound whose minimum distance is a multiple of q are q^r-divisible [67, Prop. 13].[49] Optimal codes of lengths strictly above the Griesmer bound tend to have similar divisibility properties; see, e.g., *Best Known Linear Codes* in `Magma`.

5. MORE NON-EXISTENCE RESULTS FOR q^r-DIVISIBLE SETS

For a point set $\mathscr{C}$ in $\mathrm{PG}(v-1,\mathbb{F}_q)$ let $\mathscr{T}(\mathscr{C}) := \{0 \le i \le c \mid a_i > 0\}$, where a_i denotes the number of hyperplanes with $\#(\mathscr{C}\cap H) = i$.

Lemma 20. *For integers* $u \in \mathbb{Z}$, $m \ge 0$ *and* $\Delta \ge 1$ *let* $\mathscr{C}$ *in* $\mathrm{PG}(v-1,\mathbb{F}_q)$ *be* Δ*-divisible of cardinality* $n = u + m\Delta \ge 0$. *Then, we have* $(q-1)\cdot\sum_{h\in\mathbb{Z},h\le m} h a_{u+h\Delta} = (u+m\Delta-uq)\cdot\frac{q^{v-1}}{\Delta} - m$, *where we set* $a_{u+h\Delta} = 0$ *if* $u + h\Delta < 0$.

Proof. Rewriting the equations from Lemma 6 yields $(q-1)\cdot\sum_{h\in\mathbb{Z},h\le m} a_{u+h\Delta} = q\cdot q^{v-1} - 1$ and $(q-1)\cdot\sum_{h\in\mathbb{Z},h\le m}(u+h\Delta)a_{u+h\Delta} = (u+m\Delta)(q^{v-1}-1)$. u times the first equation minus the second equation gives Δ times the stated equation. □ □

Corollary 9. *Let* $\mathscr{C}$ *in* $\mathrm{PG}(v-1,\mathbb{F}_q)$ *satisfy* $n = \#\mathscr{C} = u + m\Delta$ *and* $\mathscr{T}(\mathscr{C}) \subseteq \{u, u+\Delta, \dots, u+m\Delta\}$. *Then* $u < \frac{n}{q}$ *or* $u = n = 0$.

While the quadratic inequality of Lemma 10 is based on the first three MacWilliams identities, the linear inequality of Lemma 20 is based on the first two MacWilliams identities. Corollary 9 corresponds to the average argument that we have used in the proof of Lemma 8. Lemma 10 can of course be applied in the general case of q^r-divisible sets. First we characterize the part of the parameter space where $\tau_q(c,\Delta,m) \le 0$ and then we analyze the right side of the corresponding interval,

Lemma 21. *For* $m \in \mathbb{Z}$, *we have* $\tau_q(c,\Delta,m) \le 0$ *if and only if* $(q-1)c - (m-1/2)\Delta q + \frac{1}{2} \in$

$$\left[-\frac{1}{2}\cdot\sqrt{q^2\Delta^2 - 4qm\Delta + 2q\Delta + 1}, \frac{1}{2}\cdot\sqrt{q^2\Delta^2 - 4qm\Delta + 2q\Delta + 1}\right]. \qquad (12)$$

The last interval is non-empty, i.e., the radicand is non-negative if and only if $m \le \lfloor (q\Delta+2)/4 \rfloor$. *We have* $\tau_q(u,\Delta,1) = 0$ *if and only if* $u = (\Delta q - 1)/(q-1)$ *or* $u = 0$.

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[48] Note that $m(q-1) \equiv 0 \pmod{q^{r+1}}$ is equivalent to $m \equiv 0 \pmod{q^{r+1}}$.

[49] In the case $q = p$, and in general for codes of type BV, such codes are even q^e-divisible, where q^e is the largest power of p dividing the minimum distance [65, Th. 1 and Prop. 2].

Proof. Solving $\tau_q(c,\Delta,m)=0$ for c yields the boundaries for c stated in (12)). Inside this interval we have $\tau_q(c,\Delta,m)\le 0$. Now, $q^2\Delta^2-4qm\Delta+2q\Delta+1\ge 0$ is equivalent to $m\le \frac{q\Delta}{4}+\frac{1}{2}+\frac{1}{4q\Delta}$. Rounding downward the right-hand side, while observing $\frac{1}{4q\Delta}<\frac{1}{4}$, yields $\lfloor (q\Delta+2)/4\rfloor$. □ □

Lemma 22. *For* $1\le m\le \left\lfloor \sqrt{(q-1)q\Delta}-q+\frac{3}{2}\right\rfloor$*, we have* $(q-1)n-(m-1/2)\Delta q+\frac{1}{2}\le \frac{1}{2}\cdot\sqrt{q^2\Delta^2-4qm\Delta+2q\Delta+1}$*, where* $n=m\cdot\begin{bmatrix}r+1\\1\end{bmatrix}_q-1$ *and* $\Delta=q^r$.

Proof. Plugging in yields $\frac{1}{2}\cdot(q\Delta+3-2m-2q)\le\frac{1}{2}\sqrt{q^2\Delta^2-(4m-2)q\Delta+1}$, so that squaring and simplifying gives $m\le\sqrt{(q-1)q\Delta+1/4}-q+\frac{3}{2}$. □ □

Theorem 11. *Let* $\mathscr{C}$ *in* $\mathrm{PG}(v-1,\mathbb{F}_q)$ *be* q^1*-divisible with* $2\le n=\#\mathscr{C}\le q^2$*, then either* $n=q^2$ *or* $q+1$ *divides* n.

Proof. First we show $n\notin[(m-1)(q+1)+2,m(q+1)-1]$ for $1\le m\le q-1$. For $m=1$ this statement follows from Lemma 21 and Lemma 10. For $m\ge 2$ let $(m-1)(q+1)+2\le n\le m(q+1)-1$. Due to Lemma 10 it suffices to verify $\tau_q(n,q,m)\le 0$. From $n\ge(m-1)(q+1)+2$ we conclude

$$(q-1)n-(m-1/2)\Delta q+\frac{1}{2}\ge-\frac{1}{2}\cdot\left(q^2-4q+1+2m\right)\ge-\frac{1}{2}\cdot\left(q^2-2m-3\right)$$
$$\ge-\frac{1}{2}\cdot\sqrt{q^4-4mq^2+2q^2+1}=-\frac{1}{2}\cdot\sqrt{q^2\Delta^2-4qm\Delta+2q\Delta+1}$$

and from $n\le m(q+1)-1$ we conclude

$$(q-1)n-(m-1/2)\Delta q+\frac{1}{2}\le\frac{1}{2}\cdot\left(q^2-2m-2q+3\right)\overset{\star}{\le}\frac{1}{2}\cdot\sqrt{q^2\Delta^2-4qm\Delta+2q\Delta+1}.$$

With respect to the estimation $\star$, we remark that $-4q^3+8q^2-12q+8+4m(m+2q-3)\overset{m\le q-1}{\le}-4(q-1)(q^2-4q+6)\overset{q\ge 2}{\le}0$. Thus, Lemma 21 gives $\tau_q(n,q,m)\le 0$.

Applying Corollary 9 with $u=m$ and $\Delta=q$ yields $n\ne(m-1)(q+1)+1$ for all $1\le m\le q-1$. □ □

The existence of ovoids shows that the upper bound $n\le q^2$ is sharp in Theorem 11.

Theorem 12. *For the cardinality* n *of a* q^r*-divisible set* $\mathscr{C}$ *over* $\mathbb{F}_q$ *we have*

$$n\notin\left[(a(q-1)+b)\begin{bmatrix}r+1\\1\end{bmatrix}_q+a+1,(a(q-1)+b+1)\begin{bmatrix}r+1\\1\end{bmatrix}_q-1\right],$$

where $a,b\in\mathbb{N}_0$ *with* $b\le q-2$*,* $a\le r-1$*, and* $r\in\mathbb{N}_{>0}$.

In other words, if $n\le rq^{r+1}$*, then* n *can be written as* $a\begin{bmatrix}r+1\\1\end{bmatrix}_q+bq^{r+1}$ *for some* $a,b\in\mathbb{N}_0$.

Proof. We prove by induction on r, set $\Delta=q^r$, and write $n=(m-1)\begin{bmatrix}r+1\\1\end{bmatrix}_q+x$, where $a+1\le x\le\begin{bmatrix}r+1\\1\end{bmatrix}_q-1$ and $m-1=a(q-1)+b$ for integers $0\le b\le q-2$, $0\le a\le r-1$. The induction start $r=1$ is given by Theorem 11.

Now, assume $r\ge 2$ and conclude that for $0\le b'\le q-2$, $0\le a'\le r-2$ we have $n'\notin\left[(a'(q-1)+b')\begin{bmatrix}r\\1\end{bmatrix}_q+a'+1,(a'(q-1)+b'+1)\begin{bmatrix}r\\1\end{bmatrix}_q-1\right]$ for the cardinality n' of a q^{r-1}-divisible set. If $a\le r-2$ and $x\le\begin{bmatrix}r\\1\end{bmatrix}_q-1$, then $b'=b$, $a'=a$ yields $\mathscr{T}(\mathscr{C})\subseteq\{u,u+\Delta,\dots,u+(m-2)\Delta\}$ for $u=\Delta+(m-1)\begin{bmatrix}r\\1\end{bmatrix}_q+x$. We compute $(q-1)u=q^{r+1}-q^r+(m-1)q^r-(m-1)+(q-1)x\overset{x\ge a+1}{\ge}(m-2)q^r+q^{r+1}>(m-2)\Delta$, so that we can apply Corollary 9. If $a=r-1$ and $a+1\le x\le\begin{bmatrix}r\\1\end{bmatrix}_q-1$, then $b'=b$, $a'=a-1$ yields $\mathscr{T}(\mathscr{C})\subseteq\{u,u+\Delta,\dots,u+(m-1)\Delta\}$ for $u=(m-1)\begin{bmatrix}r\\1\end{bmatrix}_q+x$. We compute $(q-1)u=(m-1)q^r-$

$(m-1)+x(q-1) > (m-1)\Delta$ using $x \geq a+1$, so that we can apply Corollary 9. Thus, we can assume $\begin{bmatrix} r \\ 1 \end{bmatrix}_q \leq x \leq \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q - 1$ in the remaining part. Additionally we have $m \leq r(q-1)$.

We aim to apply Lemma 21. Due to Lemma 22 for the upper bound of the interval it suffices to show $r(q-1) \leq \left\lfloor \sqrt{(q-1)q\Delta} - q + \frac{3}{2} \right\rfloor$. For $q=2$ the inequality is equivalent to $r \leq \left\lfloor \sqrt{2^{r+1}} - \frac{1}{2} \right\rfloor$, which is valid for $r \geq 2$. Since the right hand side is larger then $(q-1)(\sqrt{\Delta}-1)$, it suffices to show $q^{r/2}-1 \geq r$, which is valid for $q \geq 3$ and $r \geq 2$. For the left hand side of the interval if suffices to show

$$(q-1)n - (m-1/2)\Delta q + \frac{1}{2} \geq -\frac{1}{2} \cdot \sqrt{(\Delta q)^2 - (4m-2)\Delta q + 1},$$

which can be simplified to $\Delta q + 2m - 3 - 2(q-1)x \leq \sqrt{(\Delta q)^2 - (4m-2)\Delta q + 1}$ using $n = (m-1)\begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q + x$. Since $(q-1)x \geq q^r - 1$ and $m \leq r(q-1)$ it suffices to show

$$-\Delta^2 + 2rq\Delta - 2r\Delta - \Delta - r + r^2 q - r^2 \leq 0. \tag{13}$$

For $q=2$ this inequality is equivalent to $-2^{2r} + r2^{r+1} + r^2 - 2 - 2^r \leq 0$, which is valid for $r \geq 2$. For $r=2$ Inequality (13) is equivalent to $-q^4 + 4q^3 - 4q^2 - q^2 + 4q - 6$, which is valid for $q \in \{2,3\}$ and $q \geq 4$. For $q \geq 3$ and $r \geq 3$ we have $\Delta \geq 3rq$, so that Inequality (13) is satisfied. □ □

This classification result enables us to decide the existence problem for q^r-divisible sets over $\mathbb{F}_q$ of cardinality n in many further cases. We restrict ourselves to the cases $q=2$, $r \in \{1,2,3\}$, and refer to [30] for further results.

Theorem 13. *(i)* 2^1*-divisible sets over* $\mathbb{F}_2$ *of cardinality n exist for all* $n \geq 3$ *and do not exist for* $n \in \{1,2\}$*; in particular,* $\mathrm{F}(2,1) = 2$.

(ii) 2^2*-divisible sets over* $\mathbb{F}_2$ *of cardinality n exist for* $n \in \{7,8\}$ *and all* $n \geq 14$*, and do not exist in all other cases; in particular,* $\mathrm{F}(2,2) = 13$.

(iii) 2^3*-divisible sets over* $\mathbb{F}_2$ *of cardinality n exist for*

$$n \in \{15,16,30,31,32,45,46,47,48,49,50,51\},$$

for all $n \geq 60$*, and possibly for* $n = 59$*; in all other cases they do not exist; thus* $\mathrm{F}(2,3) \in \{58,59\}$.

Proof. (i) The non-existence for $n \in \{1,2\}$ is obvious. Existence for $n \geq 3$ can be shown by taking $\mathscr{C}$ as a projective basis in $\mathrm{PG}(n-2,\mathbb{F}_q)$. The corresponding code C is the binary $[n,n-1,2]$ even-weight code.

(ii) The non-existence part follows from Theorem 12. Existence for $n \in \{7,8\}$ is shown by the $[7,3,4]$ simplex code and its dual (the $[8,4,4]$ extended Hamming code). These two examples and Lemma 15 in turn yield examples of 4-divisible point sets for $n \in \{14,15,16\}$.[50] For $n \in \{17,18,19,20\}$ Lemma 17 provides examples.[51] Together these represent all arithmetic progressions modulo 7, showing existence for $n > 20$.

(iii) Existence of 8-divisible sets for the indicated cases with $n \leq 48$ is shown in the same way as in (ii). Examples for $n=49$, $n=50$, and $n=74$ were found by computer search; we refer to [30] for generator matrices of the corresponding 8-divisible codes. The binary irreducible cyclic $[51,8]$ code, which is a two-weight code with nonzero weights 24 and 32 (see, e.g., [52]), provides an example for $n=51$.[52]

[50]The three examples are realized in dimensions 6, 7 and 8, respectively. Alternative solutions for $n \in \{15,16\}$, having smaller ambient space dimensions, are the $[15,4,8]$ simplex code and the $[16,5,8]$ first-order Reed-Muller code.

[51]These examples can be realized in $\mathbb{F}_2^6$ for $n \in \{17,18\}$ and in $\mathbb{F}_2^7$ for $n \in \{19,20\}$.

[52]It might look tempting to construct a projective 8-divisible binary code of length 50 by shortening such a code C of length 51. However, this does not work: By Lemma 24, C is the concatenation of an ovoid in $\mathrm{PG}(3,\mathbb{F}_4)$ with the binary $[3,2]$ simplex code. By construction, the corresponding 8-divisible point set $\mathscr{C}$ is the disjoint

For $n \in \{63, \ldots, 72\}$ Lemma 17 provides examples. For $n = 73$, a suitable example is given by the projective $[73, 9]$ two-weight code with non-zero weights 32 and 40 in [43]. Together with the mentioned example for $n = 74$ these represent all arithmetic progressions modulo 15, showing existence for $n > 74$.

The non-existence part follows for $n \le 48$ from Theorem 12 and for $53 \le n \le 58$ from Lemma 10 with $m = 4$. It remains to exclude an 8-divisible point set in $\mathrm{PG}(v-1, \mathbb{F}_q)$ with $\#\mathscr{C} = 52$. For this we will use a variant of the linear programming method, which treats different ambient space dimensions simultaneously. Since $\mathscr{C}$ is in particular 4-divisible, we conclude from Lemma 7 and Part (ii) that there are no 4- or 12-hyperplanes, i.e., $A_{40} = A_{48} = 0$. Using the parametrization $y = 2^{v-3}$, the first four MacWilliams identities for the associated code C are

$$\begin{array}{ccccccccccc} 1 & + & A_8 & + & A_{16} & + & A_{24} & + & A_{32} & = & 8y, \\ 52 & + & 44A_8 & + & 36A_{16} & + & 28A_{24} & + & 20A_{32} & = & 4y\cdot 52, \\ \binom{52}{2} & + & \binom{44}{2}A_8 & + & \binom{36}{2}A_{16} & + & \binom{28}{2}A_{24} & + & \binom{20}{2}A_{32} & = & 2y\cdot\binom{52}{2}, \\ \binom{52}{3} & + & \binom{44}{3}A_8 & + & \binom{36}{3}A_{16} & + & \binom{28}{3}A_{24} & + & \binom{20}{3}A_{32} & = & y\left(\binom{52}{3}+A_3^{\perp}\right). \end{array}$$

Substituting $x = yA_3^{\perp}$ and solving for $A_8, A_{16}, A_{24}, A_{32}$ yields $A_8 = -4 + \frac{1}{512}x + \frac{7}{64}y$, $A_{16} = 6 - \frac{3}{512}x - \frac{17}{64}y$, $A_{24} = -4 + \frac{3}{512}x + \frac{397}{64}y$, and $A_{32} = 1 - \frac{1}{512}x + \frac{125}{64}y$. Since $A_{16}, x \ge 0$, we have $y \le \frac{384}{17} < 23$. On the other hand, since $3A_8 + A_{16} \ge 0$, we also have $-6 + \frac{y}{16} \ge 0$, i.e., $y \ge 96$—a contradiction. □ □

The non-existence of a 2^3-divisible set of cardinality $n = 52$ implies the (compared with Theorem 4) tightened upper bound $\mathrm{A}_2(11, 8; 4) \le 132$[53] and can also be obtained from a more general result, viz., Corollary 10 with $t = 3$. Combining the first four MacWilliams identities we obtain an expression involving a certain cubic polynomial [30]:

Lemma 23. *Let $\mathscr{C}$ be Δ-divisible over $\mathbb{F}_q$ of cardinality $n > 0$ and $t \in \mathbb{Z}$. Then $\sum_{i\ge1} \Delta^2(i-t)(i-t-1)\cdot(g_1\cdot i + g_0)\cdot A_{i\Delta} + qhx = n(q-1)(n-t\Delta)(n-(t+1)\Delta)g_2$, where $g_1 = \Delta qh$, $g_0 = -n(q-1)g_2$, $g_2 = h - (2\Delta qt + \Delta q - 2nq + 2n + q - 2)$ and $h = \Delta^2q^2t^2 + \Delta^2q^2t - 2\Delta nq^2t - \Delta nq^2 + 2\Delta nqt + n^2q^2 + \Delta nq - 2n^2q + n^2 + nq - n$.*

Corollary 10. *If there exists $t \in \mathbb{Z}$, using the notation of Lemma 23, with $n/\Delta \notin [t, t+1]$, $h \ge 0$, and $g_2 < 0$, then there is no Δ-divisible set over $\mathbb{F}_q$ of cardinality n.*

Applying Corollary 10 with $t = 6$ implies the non-existence of a 2^4-divisible set $\mathscr{C}$ over $\mathbb{F}_2$ with $\#\mathscr{C} = 200$, so that $\mathrm{A}_2(16, 12; 6) \le 1032$, while Theorem 10 gives $\mathrm{A}_2(16, 12; 6) \le 1033$. There is no 8^5-divisible set $\mathscr{C}$ over $\mathbb{F}_8$ with $\#\mathscr{C} = 6\cdot 8^6 + 3 = 1572867$, which can be seen as follows. Corollary 4 implies the existence of a subspace U of co-dimension 4 such that $\mathscr{C}\cap U$ is 8^1-divisible, $\#\mathscr{C}\cap U \le 2\cdot 8^2 + 3 = 131$, and $\#\mathscr{C}\cap U \equiv 3 \pmod{8^2}$. Applying Lemma 10 with $m = 1$ and $m = 8$ excludes the existence of a 8^1-divisible set with cardinality 3 or 67, respectively. Cardinality 131 is finally excluded by Corollary 10 with $t = 14$. Thus, $\mathrm{A}_8(14; 12; 6) \le 16777237$, while Theorem 10 gives $\mathrm{A}_8(14, 12; 6) \le 16777238$ and Theorem 4 gives $\mathrm{A}_8(14, 12; 6) \le 16777248$. See also [31, 46] for a few more such examples.

Most of the currently best known bounds for $\mathrm{A}_q(v, 2k; k)$ can also be directly derived by linear programming; cf. the online tables at `http://subspacecodes.uni-bayreuth.de` [31]. The following lemma gives a glimpse on coding-theoretic arguments dealing with the MacWilliams equations and its non-negative integer solutions.

Lemma 24. *Let C be a projective 8-divisible binary code of length $n = 51$. Then C is isomorphic to the concatenation of an ovoid in $\mathrm{PG}(3, \mathbb{F}_4)$ with the binary $[3, 2]$ simplex*

union of 17 lines. In particular, each point of $\mathscr{C}$ is contained in a line in $\mathscr{C}$. Consequently, shortening C in any coordinate never gives a projective code.

[53] Consequently, for all $t \ge 2$ the upper bound for $\mathrm{A}_2(4t+3, 8; 4)$ is tightened by one; cf. Lemma 4.

code. The code C has the parameters $[51,8]$ *and the weight enumerator* $1+204X^{24}+51X^{32}$.

Proof. With the notation $k=\dim(C)$ and $y=2^{k-3}$, solving the equation system of the first three MacWilliams equations yields $A_0=1$, $A_{16}=-6-3A_8+\frac{3}{16}y$, $A_{24}=8+3A_8+\frac{49}{8}y$, and $A_{32}=-3-A_8+\frac{27}{16}y$. Since $A_{16}\geq 0$, we have $y\geq 32$ and hence $k\geq 8$. Plugging the stated equations into the fourth MacWilliams equation and solving for A_8 gives $A_8=\frac{yA_3^\perp}{512}+\frac{47y}{512}-4$ and $A_{16}=6-\frac{3yA_3^\perp}{512}-\frac{45y}{512}$. Since $A_{16}\geq 0$ and $yA_3^\perp\geq 0$, we have $6-\frac{45y}{512}\geq 0$, so that $y\leq 68+\frac{4}{15}$ and therefore $k\leq 9$.

For $k=9$, i.e., $y=64$, $A_{16}\geq 0$ gives $A_3^\perp\leq 1$. $A_3^\perp=0$ leads to $A_8=\frac{15}{8}$, which is impossible. For $A_3^\perp=1$ the resulting weight enumerator of C is $1+2X^8+406X^{24}+103X^{32}$. However, there is no such code, as the sum of the two codewords of weight 8 would be a third non-zero codeword of weight at most 16, which does not exist.

In the case $k=8$, i.e., $y=32$, the first MacWilliams equation forces $A_{16}=A_8=0$. The resulting weight enumerator of C is given by $1+204X^{24}+51X^{32}$. In particular, C is a projective $[51,8]$ two-weight code. By [11], this code is unique. The proof is concluded by the observation that an ovoid in $\mathrm{PG}(3,\mathbb{F}_4)$ is a projective quaternary $[17,4]$ two-weight code, such that the concatenation with the binary $[3,2]$ simplex code yields a projective binary $[51,8]$ two-weight code. □ □

6. Open research problems

In this closing section we have collected some open research problems within the scope of this article. All of them presumably are accessible using the theoretical framework of q^r-divisible sets. Considerably more challenging is the question whether similar methods can be developed for arbitrary constant-dimension codes in place of of partial spreads (or vector space partitions). We only mention the following example: The proof of $\mathrm{A}_2(6,4;3)=77$ still depends on extensive computer calculations providing the upper bound $\mathrm{A}_2(6,4;3)\leq 77$ [35]. The known theoretical upper bound of 81 may be sharpened to 77 (along the lines of [55]), if only the existence of a $(6,81,4;3)_2$ code can be excluded. The 81 planes in such a code would form an exact 9-cover of the line set of $\mathrm{PG}(5,\mathbb{F}_2)$.

6.1. **Better constructions for partial spreads.** The only known cases in which the construction of Theorem 2 has been surpassed are derived from the sporadic example of a partial 3-spread of cardinality 34 in $\mathbb{F}_2^8$ [21], which has 17 holes and can be used to show $\mathrm{A}_2(3m+2,6;3)\geq(2^{3m+2}-18)/7$ by adding $m-2$ layers of lifted MRD codes; cf. Lemma 4 and Corollary 2. A first step towards the understanding of the sporadic example is the classification of all 2^2-divisible point sets of cardinality 17 in $\mathrm{PG}(k-1,\mathbb{F}_2)$. It turns out that there are exactly 3 isomorphism types, one configuration $\mathcal{H}_k$ for each dimension $k\in\{6,7,8\}$. Generating matrices for the corresponding doubly-even codes are

$$\begin{pmatrix} 10000110010101110\\ 01000010111011100\\ 00100100000011000\\ 00010111001110100\\ 00001001100111110\\ 00000011100111011 \end{pmatrix},\quad \begin{pmatrix} 10000011110100110\\ 01000001111111000\\ 00100010000110000\\ 00010010000101000\\ 00001001001000100\\ 00000101001000010\\ 00000010101011111 \end{pmatrix},\quad \begin{pmatrix} 10000000111011110\\ 01000000010110000\\ 00100000011100000\\ 00010000001110000\\ 00001001100000010\\ 00000101000001010\\ 00000011000000110\\ 00000001111011101 \end{pmatrix}. \tag{14}$$

While the classification, so far, is based on computer calculations,[54] one can easily see that there are exactly three solutions of the MacWilliams identities.

[54] See `http://www.rlmiller.org/de\_codes` and [18] for the classification of, possibly non-projective, doubly-even codes over $\mathbb{F}_2^v$.

Lemma 25. *Let $\mathscr{C}$ be a 2^2-divisible point set over $\mathbb{F}_2$ of cardinality 17. Then $k = \dim\langle\mathscr{C}\rangle \in \{6,7,8\}$, and the solutions of the MacWilliams identities are unique for each k: $(k;a_5,a_9,a_{13};A_3^\perp) = (6;12,49,2;6)$, $(7;25,95,7;2)$, $(8;51,187,17;0)$.*

Proof. The unique solution of the standard equations is given by $a_5 = \frac{13}{16}\cdot 2^{k-2} - 1$, $a_9 = \frac{23}{8}\cdot 2^{k-2} + 3$, and $a_{13} = \frac{5}{16}\cdot 2^{k-2} - 3$. Hence $k \geq 6$, since otherwise $a_{13} < 0$, and $k \leq 8$, since C is self-orthogonal.[55] □ □

The hole set of the partial 3-spread in [21] corresponds to $\mathscr{H}_7$. A geometric description of this configuration is given in [47, p. 84]. We have computationally checked that indeed all three hole configurations can be realized by a partial 3-spread of cardinality 34 in $\mathbb{F}_2^8$.[56] All three configurations have a non-trivial automorphism group, and hence there is a chance to find a partial 3-spread with nontrivial symmetries and eventually discover an underlying more general construction.[57] So far we can only describe the geometric structure of $\mathscr{H}_6$, $\mathscr{H}_7$, $\mathscr{H}_8$.

The hole configuration $\mathscr{H}_6$ consists of two disjoint planes E_1, E_2 in $\mathrm{PG}(5,\mathbb{F}_2)$ and a solid S spanned by lines $L_1 \subset E_1$ and $L_2 \subset E_2$. The $17 = 4+4+9$ points of $\mathscr{H}_6$ are those in $E_1 \setminus L_1$, $E_2 \setminus L_2$, and $S \setminus (L_1 \cup L_2)$. An application of Lemma 17 (the case $q = r = i = 2$) gives that $\mathscr{H}_6$ is 4-divisible. In sunflower terms, $\mathscr{F}_1 = \{S, E_1\}$, $\mathscr{F}_2 = \{S, E_2\}$ with centers L_1, L_2, respectively.

The hole configuration $\mathscr{H}_7$ can be obtained by modifying a 3-sunflower $\mathscr{F} = \{E, S_1, S_2\}$, whose petals are a plane E and two solids S_1, S_2 and whose base is a line L. By Lemma 16(ii), the point set $E \cup S_1 \cup S_2$, of cardinality $3+4+12+12 = 31$ is 4-divisible. From this set we remove two planes $E_1 \subset S_1$, $E_2 \subset S_2$ intersecting L in different points. This gives $\mathscr{H}_7$.

The hole configuration $\mathscr{H}_8$ can be obtained by modifying the cone construction of Lemma 19. We start with a projective basis $\mathscr{B}$ in $\mathbb{F}_2^4$, i.e., $m = 5$ points with no 4 of them contained in a plane. Such point sets $\mathscr{B}$ are associated to the $[5,4,2]_2$ even-weight code and hence 2-divisible. Since $m \equiv 1 \pmod 4$, the proof of Lemma 19 shows that a generalized cone $\mathscr{K}$ over $\mathscr{B}$ with 1-dimensional vertex Q of multiplicity $-m(q-1) \equiv -m \equiv 3 \pmod 4$ is 4-divisible. Working over $\mathbb{Z}$, we can set $\chi_{\mathscr{K}}(Q) = -1$ as well. Adding any 4-divisible point set $\mathscr{D}$ containing Q (i.e., $\chi_{\mathscr{D}}(Q) = 1$) to $\mathscr{K}$ then produces a 4-divisible multiset set $\mathscr{C}$ with $\#\mathscr{C} = 10 - 1 + \#\mathscr{D}$ and $\chi_{\mathscr{C}}(Q) = 0$. By making the ambient spaces of $\mathscr{K}$ and $\mathscr{C}$ intersect only in Q (which requires embedding the configuration into a larger space $\mathbb{F}_2^v$), we can force $\mathscr{C}$ to be a set. The configuration $\mathscr{H}_8$ is obtained by choosing $\mathscr{D}$ as an affine solid.[58]

From the preceding discussion it is clear that all possible hole types of a partial 3-spread of cardinality 34 in $\mathbb{F}_2^8$ belong to infinite families of q^r-divisible sets. Can further 2^2-divisible sets of small cardinality be extended to an infinite family?

Construction 1. *For integers $r \geq 1$ and $0 \leq m \leq r$ let S be a $2r$-subspace. By $F_1, \ldots, F_{\left[{m \atop 1}\right]_q}$ we denote the $(2r+1)$-subspaces of $\mathbb{F}_{2r+m}$ that contain S and by $L_1, \ldots, L_{\left[{m \atop 1}\right]_q}$ we denote a list of r-subspaces of S with pairwise trivial intersection. Let $0 \leq a \leq \left[{m \atop 1}\right]_q$ and $0 \leq b_i \leq q^{r-1} - 1$ for all $1 \leq i \leq a$. For each $1 \leq i \leq a$ we choose $q - 1 + b_i q =: c_i$ different $(r+1)$-subspaces $E_{i,j}$ of F_i with $F_i \cap S = L_i$. With this, we set $\mathscr{C} = \left(S \setminus \cup_{i=1}^a L_i\right) \cup$*

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[55] Alternatively, the 4th MacWilliams identity yields $64 - 2^{k-2} = 2^{k-3}\cdot A_3^\perp$ and hence $k \leq 8$.

[56] 624 non-isomorphic examples can be found at `http://subspacecodes.uni-bayreuth.de` [31].

[57] In a forthcoming paper we classify the 2612 non-isomorphic partial 3-spreads of cardinality 34 in $\mathbb{F}_2^8$ that admit an automorphism group of order exactly 8, which is possible for $\mathscr{H}_6$ only, and show that the automorphism groups of all other examples have order at most 4.

[58] Since punctured affine solids are associated to the $[7,4,3]_2$ Hamming code, we may also think of $\mathscr{H}_8$ as consisting of the 2-fold repetition of the $[5,4,2]$-code and the Hamming code "glued together" in Q. In fact the doubly-even $[17,8,4]_2$ code associated with $\mathscr{H}_8$ is the code $\overline{I}_{17}^{(3)}$ in [59, p. 234]. The glueing construction is visible in the generator matrix.

$\left(\cup_{i=1}^{a} \cup_{j=1}^{c_i} (E_{i,j} \backslash L_i)\right)$ *and observe* $\dim(\mathscr{C}) \le 2r+m$, $\#\mathscr{C} = \begin{bmatrix} 2r \\ 1 \end{bmatrix}_q + a \cdot \left(q^{r+1} - \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q\right) + b \cdot q^{r+1}$*, where* $b = \sum_{i=1}^{a} b_i$*, and that* $\mathscr{C}$ *is* q^r*-divisible.*

Proof. Apply Lemma 13, 14 using $\chi_{\mathscr{C}}^{v} = \chi_{S}^{v} + \sum_{i=1}^{a} \sum_{j=1}^{c_i} \chi_{E_{i,j}}^{v} - q \sum_{i=1}^{a} (b_i+1) \chi_{L_i}^{v}$. □ □

We remark that the construction can easily be modified to obtain q^r-divisible sets of cardinality $n = \begin{bmatrix} 2r \\ 1 \end{bmatrix}_q + a \cdot \left(q^{r+1} - \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q\right) + b \cdot q^{r+1}$ and dimension v for all $2r+m \le v \le 2r + a(q-1) + bq$. Choosing $m = r$, $a = q^{r-1}$, and $b = 0$ we obtain a q^r-divisible set $\mathscr{C}$ of cardinality $n = q^{2r} + \begin{bmatrix} r-1 \\ 1 \end{bmatrix}_q$ and dimension $v = 3r$. For which parameters r and q do partial $(r+1)$-spreads of cardinality $q^{2r+1} + q^{r-1}$, with hole configuration $\mathscr{C}$, in $\mathbb{F}_q^{3r+2}$ exist? So far, such partial spreads are known for $r = 1$, $(q,r) = (2,2)$ and all further examples would be strictly larger than the ones from Theorem 2.

For the corresponding parameters over the ternary field we currently only know the bounds $244 \le \mathrm{A}_3(8,6;3) \le 248$. A putative plane spread in $\mathbb{F}_3^8$ of size 248 would have a 3^2-divisible hole configuration $\mathscr{H}$ of cardinality 56. Such a point set is unique up to isomorphism and has dimension $k = 6$. It corresponds to an optimal two-weight code with weight distribution $0^1 36^{616} 45^{112}$. The set $\mathscr{H}$ was first described by R. Hill in [32] and is known as the *Hill cap*. A generator matrix for is

$$\begin{pmatrix}
10000022110100110202111100101201021211111220012002012211 \\
01000011101210101121120010211222111210000212022200222010 \\
00100022220221020011200101120020202002111221211222001112 \\
00010010112222022102002210010101002222100222112122221200 \\
00001020121022112112001021102211121000021202220212201001 \\
00000112202002201012122002011020121222221200210020211222
\end{pmatrix}.$$

The automorphism group has order 40320. Given the large automorphism group of $\mathscr{H}$, is it possible to construct a partial plane spread in $\mathbb{F}_3^8$ with size larger than 244?

For $q = 2$, the first open case is $129 \le \mathrm{A}_2(11,8;4) \le 132$. A putative partial 4-spread of size 132 has a 2^3-divisible hole configuration of cardinality 67. Such exist for all dimensions $9 \le k \le 11$; cf. Theorem 13(iii). Can one such 2^3-divisible set be completed to a partial 4-spread?

Already the smallest open cases pose serious computational challenges. A promising approach is the prescription of automorphisms in order to reduce the computational complexity; see, e.g., [44] for an application of this so-called *Kramer-Mesner method* to constant-dimension codes. Of course, the automorphisms have to stabilize the hole configuration, whose automorphism group is known or can be easily computed in many cases.

6.2. **Existence results for q^r-divisible sets.** Even for rather small parameters q and r we cannot decide the existence question, see Table 1.

6.3. **Vector space partitions.** The most general result on the non-existence of vector space partitions (without conditions on the ambient space dimension v) seems to be:

Theorem 14. *(Theorem 1 in* [28]*) Let* $\mathscr{C}$ *be a vector space partition of type* $k^z \cdots {d_2}^b {d_1}^a$ *of* $\mathbb{F}_q^v$*, where* $a, b > 0$*.*

(i) If $q^{d_2-d_1}$ *does not divide* a *and if* $d_2 < 2d_1$*, then* $a \ge q^{d_1}+1$*;*

(ii) if $q^{d_2-d_1}$ *does not divide* a *and if* $d_2 \ge 2d_1$*, then* $a > 2q^{d_2-d_1}$ *or* d_1 *divides* d_2 *and* $a = \left(q^{d_2}-1\right) / \left(q^{d_1}-1\right)$*;*

(iii) if $q^{d_2-d_1}$ *divides* a *and* $d_2 < 2d_1$*, then* $a \ge q^{d_2} - q^{d_1} + q^{d_2-d_1}$*;*

(iv) if $q^{d_2-d_1}$ *divides* a *and* $d_2 \ge 2d_1$*, then* $a \ge q^{d_2}$*.*

For the special case $d_1 = 1$, Theorems 11 and 12, presented in Section 5, provide tighter results. For $d_1 > 1$ we can replace d_1-subspaces by the corresponding point sets and apply results for q^r-divisible sets. For vector space partitions of type $k^z \cdots 4^b 2^a$ in $\mathbb{F}_2^v$ we

q	r	n
2	3	59
2	4	130, 131, 163, 164, 165, 185, 215, 216, 232, 233, 244, 245, 246, 247
3	2	70, 77, 99, 100, 101, 102, 113, 114, 115, 128
4	2	129, 150, 151, 172, 173, 193, 194, 195, 215, 216, 217, 236, 237,238, 239, 251, 258, 259, 261, 272, 279, 280, 282, 283, 293, 301, 305, 313, 314, 322, 326, 333, 334, 335,...
5	1	40
7	1	75, 83, 91, 92, 95, 101, 102, 103, 109, 110, 111, 117, 118, 119, 125, 126, 127, 133, 134, 135, 142, 143, 151, 159, 167
8	1	93, 102, 111, 120, 121, 134, 140, 143, 149, 150, 151, 152, 158, 159, 160, 161, 167, 168, 169, 170, 176, 177, 178, 179, 185, 186, 187, 188, 196, 197, 205, 206, 214, 215, 223, 224, 232, 233, 241, 242, 250, 251
9	1	123, 133, 143, 153, 154, 175, 179, 185, 189, 195, 196, 199, 206, 207, 208, 209, 216, 217, 218, 219, 226, 227, 228, 229, 236, 237, 238, 239, 247, 248, 249, 257, 258, 259, 267, 268, 269, 277, 278, 279, 288, 289, 298, 299, 308, 309, 318, 319, 329, 339, 349, 359

TABLE 1. Undecided cases for the existence of q^r-divisible sets.

obtain 2^3-divisible sets of cardinality $n = 3a$, so that $a \in \{5, 10, 15, 16\}$ or $a \geq 20$ by Theorem13(iii). Theorem 14 gives $a = 5$ or $a \geq 9$, and $4 \mid a$ implies $a \geq 16$. However, not all results of Theorem 14 can be obtained that easy. For vector space partitions of type $k^z \cdots 4^b 3^a$ in $\mathbb{F}_2^v$ we obtain 2^3-divisible sets of cardinality $n = 7a$, giving $a = 7$ or $a \geq 9$. Theorem 14 gives $a \geq 9$, and $2 \mid a$ implies $a \geq 10$. In order to exclude $a = 7$ one has to look at hyperplane intersections in this new setting. So far, we have used $\#(H \cap \mathscr{C}) \equiv \#\mathscr{C} \pmod{q^r}$. The sets $H \cap \mathscr{C}$ have to come as a partition of type $s^d (s-1)^c$, where $c + d = a$. Here the possible values for c are restricted by the cases of q^{r-1}-divisible sets admitting the partition type $(s-1)^c$. This further reduces the possible hyperplane types, so that eventually the linear programming method can be applied. In our situation we have $\mathscr{T}(\mathscr{C}) \subseteq \{25, 49\}$, which is excluded by Lemma 20. For the general case we introduce the following notation: A point set $\mathscr{C}$ in $\operatorname{PG}(v-1, \mathbb{F}_q)$ admits the partition type $s^{m_s} \cdots 1^{m_1}$ if there exists a vector space partition of $\mathbb{F}_q^v$ of type $s^{m_s} \cdots 1^{m_1}$ that covers the points in $\mathscr{C}$ and no other points. In terms of this, we may restate the previous result as "there is no 2^3-divisible set admitting the partition type 3^7". However, we are very far from the generality and compactness of Theorem 14. Nevertheless, the sketched approach seems to be a very promising research direction (and a natural extension of the study of q^r-divisible sets).

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