The golden number and Fibonacci sequences in the design of voting structures

Josep Freixas[1]

Department of Applied Mathematics III and High Engineering School (Manresa Campus), Technical University of Catalonia, Spain. Tel.: +34-938777246, josep.freixas@upc.edu

Sascha Kurz*

Department of Mathematics, University of Bayreuth, 95440 Bayreuth, Germany. Tel.: +49-921-557353, Fax: +49-921-557352, sascha.kurz@uni-bayreuth.de

---

Abstract

Some distinguished types of voters, as vetoes, passers or nulls, as well as some others, play a significant role in voting systems because they are either the most powerful or the least powerful voters in the game independently of the measure used to evaluate power. In this paper we are concerned with the design of voting systems with at least one type of these extreme voters and with few types of equivalent voters, with this purpose in mind we enumerate these special classes of games and find out that its number always follows a Fibonacci sequence with smooth polynomial variations. As a consequence we find several families of games with the same asymptotic exponential behavior except for a multiplicative factor which is the golden number or its square. From a more general point of view, our studies are related with the design of voting structures with a predetermined importance ranking.

Key words: Game Theory, Voting systems, Complete Simple Games, Enumeration and Classification, Operational Research Structures
2000 MSC: 91A12, 91A80, 91B12

---

1. Introduction

Determining importance rankings is a significant issue in operational research. The structures which are the target of our study completely rank the items (voters or components, see e.g. [17]) from the most important to the least important according to a well-known influence relation, so that we have a hierarchy for the items. This total ranking for these structures implies also the same ranking for the most well-known measures of importance [10], [12] and [28] so that it is unchallengeable.

*Corresponding author

[1]Research partially funded by Grants SGR 2009–1029 of *Generalitat de Catalunya* and MTM 2009–08037 from the Spanish Science and Innovation Ministry. This author acknowledges the support of the Barcelona Graduate School of Economics and of the Government of Catalonia.

In order to design structures or mechanisms for a given hierarchy we need to count all the possibilities available for it. The main purpose of this paper is enumerating these versatile structures commonly used in Operations Research. Indeed, the study of ordinal preferences involves a variety of fields, including tournament theory, multiple criteria decision modeling (MCDM), and, more recently, data envelopment analysis of qualitative data. As stated in the survey by Cook in [8], the notion of voter power or relative importance has been largely ignored in studies on ordinal ranking problems, although if a tangible estimate of voter importance exists, then these voters can be treated like criteria in an MCDM problem. In fact, if a common ranking exists for the most well-known power indices, this would definitively demonstrate a given importance ordering. This approach is thus useful in designing structures ranking voters in voting institutions, workers in management enterprises or device components. Examples in these different contexts can be found in [1, 24, 30, 33, 37].

Besides this more general motivation for our studies, the paper contributes to the classification of simple games or more generally voting systems initiated in the classical monograph [32] by von Neumann and Morgenstern. Here we enumerate some classes of complete simple games, i.e. special classes of voting systems in which each player casts a "yes" or a "no" vote, and the outcome is a collective "yes" or "no" decision, with distinguished types of voters. We address our attention to complete simple games with at least one of the six types of voters: dictators, veto players, passers, null players, semi-veto players, or semi-passers, see Subsection 1.1 for the precise definitions. As far as we know the last two types have not been considered before in voting literature.

The structures under study, complete simple games, have interest in several different fields apart from voting although we adopt in this paper the standard voting background. Fields for which these structures are of interest are: circuits, clusters, threshold logic, cryptography, reliability or neural networks among others, see e.g. [41] for an overview. Recently simple games were studied using binary decision diagrams, see e.g. [3, 4].

Special types of voters in simple games were also considered in [34]. Complexity results for identifying some of the proposed distinguished types of voters can be found in [2].

1.1. The basics of simple games

In this section we briefly state the basic definitions of the examined voting systems. For more background material we refer to [41].

One of the most general class of voting systems, in which a single alternative, such as a bill or an amendment, is pitted against the status quo, is given by the class of so-called simple games.

Definition 1. A *simple game* is a pair (N, W) in which $N = \{1, 2, \ldots, n\}$ and W is a collection of subsets of N that satisfies: $N \in W$, $\emptyset \notin W$ and, the monotonicity property, if $S \in W$ and $S \subseteq T \subseteq N$ then $T \in W$.

The set N is called the *grand coalition*. Members of N are called *players* or *voters*, and subsets of N are called *coalitions*, the coalitions that belong to W are called *winning coalitions*. The subfamily of *minimal* winning coalitions $W^m = \{S \in W : \forall T \subset S \Rightarrow T \notin W\}$ determines the game. The subsets of N that are in $2^N \setminus W$ are called *losing coalitions*. By $|S|$ we mean the cardinality of a coalition S. Real–world examples of simple games are given in [30, 39, 41] among others.

Example 1. (i) In the former USSR the three top state officials, the President, the Prime minister, and the Minister of Defence (Ustinov, Brezhnev, Kosygin), all had "nuclear suitcases". Any two of them could authorize a launch of a nuclear warhead. No one could do it alone.[2] This situation can be modeled as a simple game with grand coalition $N = \{1, 2, 3\}$ and where the set of winning coalitions is given by $W = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Thus, the set of minimal winning coalitions is given by $W^m = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

(ii) Let (N, W) with $N = \{1, 2, 3, 4, 5\}$ and $W = \{S \subseteq N \,:\, |S| \geq 3,\ S \neq \{3, 4, 5\}\}$. It is easy to check that $W^m = \{S \subseteq N \,:\, |S| = 3,\ S \neq \{3, 4, 5\}\}$.

The monotonicity property of simple games is some kind of a universal base assumption for voting systems. Most of the systems used in practice satisfy additional requirements. One common idea is the concept of influence, i.e. that a particular voting system may give one voter more influence than another. The so-called "desirability" relation defined on the set of voters represents a way to make this precise. Isbell already used it in [23].

Definition 2. Let (N, W) be a simple game.

(i) Player i is *at least as desirable* as j ($i \succsim j$, in short) in (N, W) if:

$$S \cup \{j\} \in W \;\Rightarrow\; S \cup \{i\} \in W, \qquad \text{for all } \; S \subseteq N \setminus \{i, j\}.$$

(ii) Players i and j are *equally desirable* ($i \approx j$, in short) in (N, W) if: $i \succsim j$ and $j \succsim i$.

(iii) Player i is *strictly more desirable* than player j ($i \succ j$, in short) in (N, W) if: $i \succsim j$ and $i \not\approx j$.

In Example 1-(i) players 1, 2 and 3 are equally desirable. In Example 1-(ii) players 1 and 2 are equally desirable, players 3, 4 and 5 are equally desirable and player i is strictly more desirable than player j for all $i = 1, 2$ and $j = 3, 4, 5$. The $\approx$-relation partitions the set of voters into equivalence classes N_i. We say that the voters in the same equivalence class have the same influence and belong to the same *type* (of voters). We also speak of the number of types, t, of voters meaning the number of equivalence classes. E.g. $t = 1$ in Example 1-(i), whereas we have $t = 2$ in Example 1-(ii).

Definition 3. A simple game (N, W) is *complete* or *linear* if the desirability relation is a complete preordering.

From now on we only consider complete simple games, abbreviated complete games, and w.l.o.g. we assume $1 \succsim \cdots \succsim n$ in the following, i.e. completeness of the desirability relation on (N, W) with player 1 being the strongest and player n being the weakest by the desirability relation.

The two voting systems described in Example 1 are complete games and we have: $1 \approx 2 \approx 3$ in Example 1-(i) and $1 \approx 2 \succ 3 \approx 4 \approx 5$ in Example 1-(ii). In this latter case N decomposes into the two equivalence classes: $N_1 = \{1, 2\}$ and $N_2 = \{3, 4, 5\}$.

[2]Example taken from the presentation "Secret sharing schemes and complete simple games" by Arkadii Slinko.

Example 2. A simple game (N, W) is a *weighted game* if there exists a vector of *weights* with non-negative components $w = (w_1, \ldots, w_n)$ such that

$$\sum_{i \in S} w_i > \sum_{j \in T} w_j, \qquad \text{for all } S \in W \text{ and } T \notin W.$$

All weighted games are complete, because $w_i \geq w_j$ implies $i \succsim j$. We have $w_i > w_j$ if $i \succ j$, but not necessarily $w_i = w_j$ if $i \approx j$.

In a weighted game a *quota* q may be inserted between $b = \min\limits_{S \in W} \sum\limits_{i \in S} w_i$ and $a = \max\limits_{T \notin W} \sum\limits_{j \in T} w_j$, i.e. any $q \in (a, b]$ separates the weights of winning coalitions from the weights of losing coalitions. As short-hand notation we use $[q; w_1, \ldots, w_n]$ for an arbitrary representation of a weighted game. There is an infinite number of representations for a weighted game with different weights, because $[q; w_1, \ldots, w_n]$ is equivalent to $[c \cdot q; c \cdot w_1, \ldots, c \cdot w_n]$ for all $c > 0$.

The voting system from Example 1-(i) is weighted and a representation for it is given by $[2; 1, 1, 1]$. The voting system from Example 1-(ii) is weighted and a representation for it is given by $[7; 3, 3, 2, 2, 2]$. For more than 5 voters one can easily find examples of complete games that are not weighted. Many of the real-world voting systems are weighted games (or at least can be represented as the intersection of two or three weighted games). We remark that every simple game is the intersection of a finite number, the minimal number is called its *dimension*, of weighted games, see [41] and [15] for some results on the dimension of complete games and simple games.

Most well-known voting systems in use are complete:

Example 3. (i) A system to amend the Canadian Constitution used in the sixties and studied in Kilgour [25] is an example of a complete game with two types of voters. In this example: $|N| = 10$, the elements of N are the ten Canadian provinces and $W^m = \{S \subseteq N : |S| = 7 \text{ and } S \cap \{1, 2\} \neq \emptyset\}$. It is then obvious N decomposes into $N_1 = \{1, 2\}$, representing the provinces of Ontario and Quebec, and $N_2 = \{3, \ldots, 10\}$. This voting system is complete but not weighted and has dimension 2, see [39] for details.

(ii) A typical example of an important complete voting system is the United Nations Security Council. The voters in this system are the fifteen countries that make up the Security Council, five of which are permanent members whereas the other ten are non-permanent members. Passage requires a total of at least nine of the fifteen possible votes, subject to a veto due to a nay vote from any one of the five permanent members. In this example: $|N| = 15$, $W^m = \{S \subseteq N : |S| = 9 \text{ and } \{1, 2, 3, 4, 5\} \subseteq S\}$. It is then obvious that N decomposes into $N_1 = \{1, 2, 3, 4, 5\}$, i.e. the five permanent nations in the Council: USA, China, France, Russia and the United Kingdom, and $N_2 = \{6, \ldots, 15\}$. The elements of N_1 are exactly the veto players. This voting system is weighted with representation:

$$[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

This model ignores abstention, for a treatment of this example considering the possibility of abstention we refer the reader to [18].

(iii) All voting systems successively used by the European Economic Community since 1958 are complete. As an example we mention the early European Economic Community between 1958 and 1973. The six founders were West Germany, France, Italy, the Netherlands, Belgium, and Luxembourg. At that time the voting procedure was applied by using the following representation as a weighted game $[12; 4, 4, 4, 2, 2, 1]$, which is equivalent to $[6; 2, 2, 2, 1, 1, 0]$. See e.g. [29] for an analysis of all the voting systems for the EU councils of ministers up to now.

An interest of complete games has emerged recently in the field of Cryptography. Indeed, the access structure in a secret sharing (see e.g., Stinson [36]) can also be modeled by a simple game. To this end Simmons [35] introduced the concept of a hierarchical access structure. Gvozdeva et al. [19] study complete games with only one shift-minimal winning coalition (see next definition) and observe that they are isomorphic concepts to conjunctive and disjunctive (for the dual game) hierarchically access structures. Moreover, both conjunctive and disjunctive hierarchically access structures have been proved to be ideal (Tassa [38]) which means they can carry the most informationally efficient secret sharing scheme and be completely secure (i.e., not giving any information about the secret to unauthorized coalitions).

Definition 4. In a simple game (N, W) a coalition S is *shift-minimal* winning if $S \in W^m$ and $(S \setminus \{i\}) \cup \{j\} \notin W$ for all $i \in S$ and $j \notin S$ with $i \succ j$. Let W^s denote the set of shift-minimal winning coalitions.

Note that a winning coalition can be minimal but not necessarily shift-minimal. E.g. in Example 1-(ii) $W^m \setminus W^s = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\}$ and $W^m \setminus W^s = \{S \cup \{1, 2\}$ with $|S| = 5$ and $S \subseteq N \setminus \{1, 2\}\}$ for the voting system to amend the Canadian Constitution described in [25]. For the remaining examples described above we have $W^m = W^s$.

While most of the common voting systems consist of many types of equivalent voters, examples with only few types of voters are not that artificial as one might think at the first moment. As the structural properties of voting systems with few types of voters seem to be more accessible, they are also a good starting point for theoretical considerations. The enumeration of such voting structures is indeed the target of our study. In the field of Boolean algebra, complete games correspond to 2-monotonic positive Boolean functions, which were already considered in [22]. The problem of identifying this type of functions by using polynomial-time recognition has been treated in [5, 6].

We introduce some distinguished types of voters for simple games.

Definition 5. Let (N, W) be a simple game.

i) $i \in N$ is a *dictator* in (N, W) if $W^m = \big\{\{i\}\big\}$,

ii) $i \in N$ has *veto* in (N, W) if $i \in S$ for all $S \in W$,

iii) $i \in N$ is a *passer* in (N, W) if $\{i\} \in W$,

iv) $i \in N$ is *null* in (N, W) if $i \notin S$ for all $S \in W^m$,

v) $i \in N$ has *semi-veto* in (N, W) if $N \setminus \{i\} \in W$ and $S \in W$ implies either $S = N \setminus \{i\}$ or $i \in S$.

vi) $i \in N$ is a *semi-passer* in (N, W) if $\{i, j\} \in W$ for all $j \in N \setminus \{i\}$, but $\{i\} \notin W$.

All three players in the complete game of Example 1-(i) are both semi-vetoes and semi-passers. Permanent nations of Example 1-(ii) have veto right and Luxembourg in Example 1-(iii) is a null player.

If a game has a dictator, the dictator is the unique player with this status and the remaining players are null voters. Being a dictator is the most radical form of having veto and of being a passer. A player have veto and is a passer *if and only if* the game is the dictatorship of this player. Thus, veto and passers are pairwise incompatible in the same game if this is not the dictatorship of a voter. If some of these types of voters are present in the game, then they form an equivalence class N_1 whose members dominate by the desirability relation *all* the other players in N. On the other hand, it is obvious that if $n = 1$ there cannot be null voters; if $n > 1$ and the game has null voters they form an equivalence class N_t whose members are dominated by the desirability relation for any other player in N.

It is also obvious from Definition 5 that veto and semi-veto (or passer and semi-passer) voters can concur in the same game, while veto and semi-passer or passer and semi-veto cannot concur in the same game (if we assume that their roles are taken by different players). Semi-vetoes belong to the same equivalence class N_1 which is the strongest one by the desirability relation if the game has no veto players, while they belong to the second class N_2 if the game has veto players, and similarly for semi-passers and passers. Finally, the concurrence of semi-vetoes and nulls in the same game, and similarly, the concurrence of semi-passers and nulls in the same game is not possible.

In summary, if a game has either a dictator, veto players or passers they are the strongest players in the game and belong to the most powerful equivalence class; if a game has null voters they are the weakest players in the game and belong to the least powerful class of the game.[3] If a game has veto and semi-veto players or passers and semi-passers, the semi-vetoes and semi-passers are the second strongest players in the game, while in the absence of veto and passers they are the most strongest players in the game.

Definition 6. The dual game (N, W^*) of a simple game (N, W) is defined by $W^* = \{S \subseteq N : N \setminus S \notin W\}$.

Hence, to win in the dual game is to block in the original one (S is *blocking* in (N, W) if $N \setminus S \notin W$). It is easy to verify that: $(W^*)^* = W$, $\succ^* = \succ$ and $\approx^* = \approx$, thus $\succsim^* = \succsim$, and a game is complete *if and only if* the dual is.

From the definition of dual game, it easily follows that:

1. If (N, W) is the dictatorship of player i, then $(N, W) = (N, W^*)$;
2. Voter $i \in N$ has veto in (N, W) *if and only if* voter $i \in N$ is a passer in (N, W^*);
3. Voter $i \in N$ is null in (N, W) *if and only if* voter $i \in N$ is null in (N, W^*);
4. Voter $i \in N$ has semi-veto in (N, W) *if and only if* voter $i \in N$ is a semi-passer in (N, W^*).

[3] Almost all power indices respect the desirability relation, e.g. the Shapley-Shubik, Banzhaf or Johnston indices [10], [12], and [28]. Thus belonging to the strongest class in the game means being one of the most powerful players, and belonging to the weakest class in the game means being one of the least powerful players in the game.

1.2. Related work

One of the first and most prominent enumeration results on simple games, a super class of complete games, is *May's Theorem for Simple Games* [31] stating that the number of simple games with n voters for simple majority decisions equals one[4]. If only anonymous or symmetric (i.e. any pair of voters are equally desirable) voters are considered for simple games with n voters we get

$$SG(n,1) = CG(n,1) = WG(n,1) = n.$$

Here $SG(n,t)$ denotes the number of simple games, $CG(n,t)$ the number of complete games, and $WG(n,t)$ the number of weighted voting games with n voters from t different types of equivalent voters.

The number $CG(n,2)$ of complete games with n voters belonging to exactly two types of voters were recently enumerated in [14] and later on in [27], giving a simpler proof:

$$CG(n,2) = F(n+6) - (n^2+4n+8) \in \Theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right), \tag{1}$$

where $F(n)$ are the Fibonacci numbers which constitute a well–known sequence of integer numbers defined by the following recurrence relation: $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for all $n > 1$.

By a *type of shift-minimal winning* coalitions we mean a set of shift-minimal winning coalitions such that any coalition in the set can be converted into any other coalition in the set by only swapping equally desirable players. The number of complete games with one shift-minimal winning coalitions, see e.g. [7, 19], was determined in [16]: $\sum_{t=1}^{n} CG(n,t,1) = 2^n - 1$, where $CG(n,t,r)$ denotes the number of complete games with n voters, t equivalent types of voters, and r shift-minimal winning coalitions. For complete games with two shift-minimal winning coalitions a more complicated enumeration formula was determined in [27]. For given values of the parameters t and r it is possible to compute an exact enumeration formula for $CG(n,t,r)$ based on the parametric Barvinok algorithm and a tailored decomposition of a certain linear programming formulation for complete games, see [27]. We remark that the exact numbers of simple games are known up to $n = 8$ voters and the exact number of complete games or weighted games are known up to $n = 9$ voters, see e.g. [26, 27].

1.3. Our contribution

We establish bijections among several classes of complete games containing at least one of the mentioned distinguished types of players, and obtain exact enumerations for these games with less than four types of equivalent voters and for four types whenever null voters are present with either veto players or passers. While these enumerations are polynomial for only two types of voters, they follow a Fibonacci sequence modified by a polynomial expression. So Fibonacci sequences and the golden number are in the core of these enumerations. The obtained sequences in this paper have not yet appeared in the On-line Encyclopedia of Integer Sequences (http://oeis.org/).

[4]May originally considered a slightly different setting of anonymous voting systems for two alternatives that are *neutral*, i.e. there is no built-in bias towards "yes" or "no" outcomes.

1.4. Organization of the paper

In Subsection 1.1 we briefly recalled some necessary background on simple games, complete games, i.e. the examined voting systems, and some types of players. To get the results in the present paper we will make use of one additional previous result, besides the enumeration of $CG(n, 2)$, namely a parametrization for complete games given in [7], which we will restate in Section 2. In Section 3 we prove that four classes of complete games with voters being very powerful or being nulls have all the same cardinality, additionally a closed formula on the number of voters is given for the number of these games as long as the number of types of voters is less than 4. The main tool to that end is Theorem 2 stating bijections between complete games containing at least one of the distinguished types of voters. In Section 4 we consider the case of complete games containing at least two of the distinguished types of voters. Finally, we end with a conclusion and some remarks on future research in Section 6.

2. Characteristic invariants of complete simple games

In this section we mainly describe the parametrization in [7] that will be used in the remaining of the paper. The relations $\succsim$ and $\approx$ can be easily generalized to coalitions:

- $S \approx_c T$ *if and only if* S may be transformed into T by only changing equally desirable players, and
- $S \succsim_c T$ *if and only if* $T = \emptyset$ or for each player $j \in T$ there is a (distinct)[5] player $i \in S$ with $i \succsim j$.

Relations $\approx_c$ and $\succsim_c$ for coalitions are, respectively, extensions of $\approx$ and $\succsim$ in the sense that, for example, $i \succsim j$ *if and only if* $\{i\} \succsim_c \{j\}$. We remark that $\approx_c$ can occur for coalitions of the same size only, while $\succsim_c$ can compare coalitions of different sizes. Moreover, $\approx_c$ is the equivalence relation associated with $\succsim_c$, and therefore $\succsim_c$ induces an ordering $\succeq$ in the quotient set $\overline{2^N} = 2^N / \approx_c$. The $\approx_c$-class of a coalition $S \subseteq N$ will be denoted by $\overline{S}$. Observe that if: $S \approx_c T$, then:

1. $S \in W^s$ *if and only if* $T \in W^s$,
2. $S \in W^m$ *if and only if* $T \in W^m$, and
3. $S \in W$ *if and only if* $T \in W$.

If $T \succsim_c S$ and $S \in W$, then $T \in W$.

If (N, W) is a simple game and $N_1, \ldots, N_t$ are the $\approx_c$-classes of the game and have cardinalities $n_1, \ldots, n_t$. Then we have:

(a) $S \approx_c T$ *if and only if* $|S \cap N_k| = |T \cap N_k|$ for all k.
(b) For every $\overline{S} \in \overline{2^N}$, the numbers $s_k = |S \cap N_k|$ do not depend on the representative S, and satisfy $0 \leq s_k \leq n_k$ for all k.
(c) Conversely, any vector $\overline{s} = (s_1, \ldots, s_t)$ such that $0 \leq s_k \leq n_k$ for all k defines a unique $\approx_c$-class $\overline{S} \in \overline{2^N}$.

[5] i.e., once player $i \in S$ is used for $j \in T$ it cannot be used again for any other player $k \in T \setminus \{j\}$.

The vector $\overline{s}$ is called the *vector of indices* associated with $\overline{S}$: it provides the common model, in terms of equally desirable players, of all coalitions belonging to $\overline{S}$. For instance, in Example 1-(i), the vector of indices (2) is associated with $\overline{\{1,2\}}$, where $\overline{\{1,2\}} = \{\{1,2\},\{1,3\},\{2,3\}\}$.

$\mathbb{N}$ will denote the set of natural numbers. If $\overline{n} \in \mathbb{N}^t$, we consider $\Lambda(\overline{n}) = \{\overline{s} \in (\mathbb{N} \cup \{0\})^t \ : \overline{n} \geq \overline{s}\}$, where $\geq$ stands for the ordinary componentwise ordering, i.e. $\overline{s} \geq \overline{r}$ *if and only if* $s_k \geq r_k$ for $k = 1, \ldots, t$. Thus, if $\overline{n} = (n_1, \ldots, n_t)$, $\Lambda(\overline{n})$ consists of all vectors $\overline{s} = (s_1, \ldots, s_t)$ whose components are integer and satisfy $0 \leq s_k \leq n_k$ for all k. We need to consider not only the ordering $\geq$, but also the weaker ordering δ given by comparison of partial sums; that is,

$$\overline{s}\,\delta\,\overline{r} \quad \textit{if and only if} \quad \sum_{i=1}^{k} s_i \geq \sum_{i=1}^{k} r_i \text{ for } 1 \leq k \leq t.$$

If $\overline{s}\,\delta\,\overline{r}$ it is said that $\overline{s}$ *dominates* $\overline{r}$. For instance, in Example 1-(ii) we have

$$\Lambda((2,3)) = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(2,3)\}.$$

If (N,W) is a complete game, following Theorem 3.1 in [7], let $N_1 > N_2 > \cdots > N_t$ be the linear ordering of the $\approx$-classes and let $\overline{n} = (n_1, n_2, \ldots, n_t)$ be the vector defined by their cardinalities. Then the map $\psi : (\overline{2^N}, \succeq) \to (\Lambda(\overline{n}), \delta)$, which assigns to each $\approx$-class $\overline{S}$ the vector of indices $\overline{s} = (s_1, s_2, \ldots, s_t)$ is an isomorphism of ordered sets.

The pair $(\Lambda(\overline{n}), \delta)$ is called the *lattice associated with the complete game* (N,W), and says that $\overline{s} = \psi(\overline{S})$ is the *model* of coalition S. The sets

$$\overline{W} = \{\overline{S} \in \overline{2^N} \,:\, S \in W\}, \quad \overline{W^m} = \{\overline{S} \in \overline{2^N} \,:\, S \in W^m\}, \quad \overline{W^s} = \{\overline{S} \in \overline{2^N} \,:\, S \in W^s\}.$$

are well defined and contain respectively, the classes of winning, minimal winning and shift-minimal winning coalitions[6] of the game. For a complete game, $\overline{W^s}$ gives enough information to reconstruct the game. For instance in Example 1-(ii) we have

$$\psi(\overline{W}) = \{(1,2),(1,3),(2,1),(2,2),(2,3)\}, \quad \psi(\overline{W^m}) = \{(1,2),(2,1)\}, \quad \psi(\overline{W^s}) = \{(1,2)\}.$$

Two simple games are essentially the same if they only vary by a change (formally a permutation) in the names assigned to players.

Definition 7. Two simple games (N,W) and (N',W') are said to be *isomorphic* if there is a bijective map $f : N \to N'$ such that $S \in W$ *if and only if* $f(S) \in W'$; f is called an isomorphism of simple games.

In the following we restate a known parametrization result of complete games up to isomorphism, which has three parts. The first part shows how to associate a vector $\overline{n}$ and a matrix $\mathcal{M}$ to a complete game (N,W) and describes the restrictions that these parameters need to fulfill. The second part establishes that isomorphic complete games (N,W) and (N',W') correspond to the same associated vector $\overline{n}$ and matrix $\mathcal{M}$ (uniqueness). The third part shows that a vector $\overline{n}$ and a matrix $\mathcal{M}$ fulfilling the conditions in Part A correspond to a complete game (N,W) (existence).

[6] In [7] the shift-minimal winning coalitions are called δ-minimal winning coalitions, because they are minimal winning for relation δ in $\Lambda(\overline{n})$.

Theorem 1. *(Carreras and Freixas' Theorems 4.1 and 4.2 in [7]) Let* (N, W) *be a complete game with* t *equivalence* $\approx$*-classes of voters* $N_1 > \cdots > N_t$*, and let* $\overline{n} = (n_1, \ldots, n_t)$ *be the vector defined by their cardinalities. Let*

$$\mathcal{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & \ldots & m_{1,t} \\ \vdots & \vdots & \ddots & \vdots \\ m_{r,1} & m_{r,2} & \ldots & m_{r,t} \end{pmatrix}$$

be the matrix whose rows $\overline{m}_p = (m_{p,1}, \ldots, m_{p,t})$ *are the models of shift-minimal winning coalitions, i.e. the vectors of indices associated to the* $\approx_c$*-classes* $\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_r$ *found in* $\overline{W^s}$*. We also assume in what follows that the rows (if* $r \geq 2$*) of* $\mathcal{M}$ *are lexicographically ordered by partial sums, i.e. if* $p < q$*, then there exists some minimum element* k *(*$1 \leq k \leq t-1$*) such that* $m_{p,k} > m_{q,k}$ *and* $m_{p,i} = m_{q,i}$ *for all* $i < k$*. The pair* $\overline{n}, \mathcal{M})$ *is called the characteristic invariant of the complete simple game* (N, W)*.*

Part A *The vector* $\overline{n}$ *and the matrix* $\mathcal{M}$ *associated with a complete game* (N, W) *satisfy the following properties:*

1. $n_k > 0$ *for all* $k = 1, \ldots, t$*;*
2. $\overline{0} \leq \overline{m}_p \leq \overline{n}$ *for* $p = 1, \ldots, r$*, and* $m_{1,1} > 0$
3. $\overline{m}_p$ *and* $\overline{m}_q$ *are not* δ*-comparable for all* $p \neq q$*; and*
4. *if* $t > 1$ *then for every* $k < t$ *there exists some* p *such that*[7]

$$m_{p,k} > 0, \;\; m_{p,k+1} < n_{k+1}.$$

Part B *(Uniqueness) Two complete games* (N, W) *and* (N', W') *are isomorphic* if and only if $\overline{n} = \overline{n'}$ *and* $\mathcal{M} = \mathcal{M}'$*.*

Part C *(Existence) Given a vector* $\overline{n}$ *and a matrix* $\mathcal{M}$ *satisfying the conditions of Part A, there exists a complete game* (N, W) *the characteristic invariants of which are* $\overline{n}$ *and* $\mathcal{M}$*.*

Example 4. (Examples 1 and 3 revisited) From the equivalence classes, its total ordering and the shift-minimal winning coalitions, we may easily derive the characteristic invariants for all the games considered above.

(a) The system employed in the former USSR for an hypothetical launch of a nuclear warhead, the game considered in Example 1-(i), may be characterized by $\overline{n} = (3)$ and $\mathcal{M} = (2)$.
(b) The game considered in Example 1-(ii) may be characterized by $\overline{n} = (2, 3)$ and $\mathcal{M} = (1\ 2)$.
(c) The voting system to amend the Canadian Constitution used in the sixties, see Example 3-(i), may be simply described by $\overline{n} = (2, 8)$ and $\mathcal{M} = (1\ 6)$.
(d) The voting system of the United Nations Organization, Example 3-(ii), may be simply described by $\overline{n} = (5, 10)$ and $\mathcal{M} = (5\ 4)$.

[7]The lexicographic ordering chosen guarantees uniqueness under permutation of rows. This lexicographic ordering is a plausible election which could be replaced for other alternative criteria.

(e) The voting systems for the early European Economic Community between 1958 and 1973 considered in Example 3-(iii) may be simply described by $\overline{n} = (3, 2, 1)$ and $\mathcal{M} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix}$.

Theorem 1 is a *parametrization* theorem because it allows one to enumerate all complete games up to isomorphism by listing the possible values of certain invariants. Based on this it is possible to compute an explicit enumeration formula for the number of complete games with n voters if the parameters t and r, i.e. the number of columns and rows of $\mathcal{M}$, are specified (but arbitrary), see [27]. This paper goes more deeply into the issue of enumerations for special cases without assuming information on r.

2.1. Special forms of the characteristic invariants of complete simple games

From now on we identify complete games with their characteristic invariants. This convention will be used in the remainder of the paper.

If a complete game has special types of voters, as those introduced in Definition 5, then the characteristic invariants, $(\overline{n}, \mathcal{M})$, that define it (and therefore fulfill the properties in Theorem 1-(A)) have specific forms. In the next seven lemmas, which will be intensively used in the following section, we consider the specific form of the characteristic invariants for complete games. Recall that $\sum_{i=1}^{t} n_i = n$ hereafter.

The proofs of the next lemmas are left to the reader, they all follow from Definition 5 and from conditions of part A in Theorem 1. After stating them, we proceed to prove just one of them: Lemma 5. The proof of Lemma 6 is similar to the proof of Lemma 5. The proofs for the other lemmas are shorter and easier.

Lemma 1. *If the game has a dictator then* $t = 2$, $\overline{n} = (1, n-1)$ *and matrix* $\mathcal{M}$ *has the form:* $\mathcal{M} = (1\ 0)$.

Note that the set of minimal winning coalitions for the game is $W^m = \{\{1\}\}$ independently of the number of players.

Lemma 2. *If the game has* k *veto players then* $k = n_1$ *and matrix* $\mathcal{M}$ *has the form:* $\mathcal{M} = (n)$ *if* $t = 1$, $\mathcal{M} = (n_1\ a)$ *with* $a < n_2$ *if* $t = 2$, *and*

$$\mathcal{M} = \begin{pmatrix} n_1 & m_{1,2} & \dots & m_{1,t} \\ n_1 & m_{2,2} & \dots & m_{2,t} \\ \vdots & \vdots & \ddots & \vdots \\ n_1 & m_{r,2} & \dots & m_{r,t} \end{pmatrix} \tag{2}$$

with $m_{1,2} > 0$ *if* $t > 2$.

Note that if $t = 1$, the game is the unanimity game given by $W = \{\{1, 2, \dots, n\}\}$. If $t = 2$, the game is $W^m = \{N_1 \cup S,$ for all $S \subseteq N_2$ and $|S| = a\}$ for some $0 \le a < n_2$.

Lemma 3. *If the game has* k *passers then* $k = n_1$ *and matrix* $\mathcal{M}$ *has the form:* $\mathcal{M} = (1)$ *if* $t = 1$,

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$$

with $n_2 > 1$ and $b > 1$ if $t = 2$, and

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & m_{2,2} & \dots & m_{2,t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & m_{r,2} & \dots & m_{r,t} \end{pmatrix} \quad (3)$$

with $m_{2,2} > 0$ if $t > 2$.

Note that if $t = 1$, the game is given by $W^m = \{\{1\}, \dots, \{n\}\}$. If $t = 2$, the game is given by $W^m = \{\{1\} \cup S, \text{ for all } S \subseteq N_2 \text{ and } |S| = b\}$ for some $0 < b \leq n_2$.

Lemma 4. *If the game has k null voters then $t > 1$, $k = n_t$ and matrix $\mathcal{M}$ has the form: $\mathcal{M} = (c\ 0)$ for some $c > 0$ if $t = 2$, and*

$$\mathcal{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,t-1} & 0 \\ m_{2,1} & m_{2,2} & \dots & m_{2,t-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{r,1} & m_{r,2} & \dots & m_{r,t-1} & 0 \end{pmatrix}$$

if $t > 2$.

Note that if $t = 2$, the game is given by $W^m = \{S \subseteq N_1 : |S| = c\}$ for some $0 < c \leq n_1$.

Lemma 5. *If the game has k_1 vetoes and k_2 semi-vetoes then $k_1 = n_1$, $k_2 = n_2$ and matrix $\mathcal{M}$ has the form: $\mathcal{M} = (n_1\ n_2 - 1)$ if $t = 2$, the form*

$$\mathcal{M} = \begin{pmatrix} n_1 & n_2 & c \\ n_1 & n_2 - 1 & n_3 \end{pmatrix}$$

with $n_3 > 1$ and $c < n_3 - 1$ if $t = 3$, and the form

$$\mathcal{M} = \begin{pmatrix} n_1 & n_2 & m_{1,3} & \dots & m_{1,t-1} & m_{1,t} \\ n_1 & n_2 & m_{2,3} & \dots & m_{2,t-1} & m_{2,t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n_1 & n_2 & m_{r-1,3} & \dots & m_{r-1,t-1} & m_{r-1,t} \\ n_1 & n_2 - 1 & n_3 & \dots & n_{t-1} & n_t \end{pmatrix}$$

if $t > 3$.

If the game has no vetoes but k semi-vetoes then $k = n_1$ and matrix $\mathcal{M}$ has the form: $\mathcal{M} = (n - 1)$ if $t = 1$,

$$\mathcal{M} = \begin{pmatrix} n_1 & a \\ n_1 - 1 & n_2 \end{pmatrix}$$

with $n_2 > 1$ and $a < n_2 - 1$ if $t = 2$, and the form

$$\mathcal{M} = \begin{pmatrix} n_1 & m_{1,2} & \dots & m_{1,t} \\ n_1 & m_{2,2} & \dots & m_{2,t} \\ \vdots & \vdots & \ddots & \vdots \\ n_1 & m_{r-1,2} & \dots & m_{r-1,t} \\ n_1 - 1 & n_2 & \cdots & n_t \end{pmatrix}$$

if $t > 2$.

Note that if $t = 2$ the game is given by $W^m = \bigcup_{i \in N_2} N \setminus \{i\}$. If $t = 3$, the game is given by $W^m = \{N \setminus \{i\} \ : \ \text{for all } i \in N \setminus N_1\} \cup \{N \setminus S \ : \ S \subseteq N_3, \ |S| = n_3 - c\}$ for some $0 \leq c < n_3 - 1$.

Lemma 6. *If the game has k_1 passers and k_2 semi-passers then $k_1 = n_1$, $k_2 = n_2$ and matrix $\mathcal{M}$ has the form:*

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

if $t = 2$ and $n_2 = 1$ or

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

if $t = 2$ and $n_2 > 1$,

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

with $n_3 > 1$ if $t = 3$ or

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & b \end{pmatrix}$$

with $b > 2$ and $n_3 > 2$ if $t = 3$, and

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & m_{3,3} & \dots & m_{3,t-1} & m_{3,t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & m_{r,3} & \dots & m_{r,t-1} & m_{r,t} \end{pmatrix}$$

if $t > 3$.

Note that if $t = 2$ and $n_2 = 1$, the game is given by $W^m = \{\{i\}, \ i \in N \setminus \{n\}\}$. If $t = 2$ and $n_2 > 1$ the game is given by $W^m = \{\{i\}, \ i \in N_1\} \cup \{\{j, k\}, \ j, k \in N_2, \ j \neq k\}$.

Lemma 7. *If the game has no passers but k semi-passers then $k = n_1$ and matrix $\mathcal{M}$ has the form: $\mathcal{M} = (2)$ if $t = 1$,*

$$\begin{pmatrix} 1 & 1 \\ 0 & c \end{pmatrix}$$

with $n_2 > 2$ and $c \geq 3$ if $t = 2$, and the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & m_{2,2} & \dots & m_{2,t-1} & m_{2,t} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & m_{r,2} & \cdots & m_{r,t-1} & m_{r,t} \end{pmatrix}$$

if $t > 2$.

Note that if $t = 1$, we have the game given by $W^m = \{\{i, j\}, \ i, j \in N, \ i \neq j\}$. If $t = 2$, we have that the game is given by $W^m = \{\{i, j\}, \ i \in N_1, \ j \in N, \ i \neq j\} \cup \{S \subseteq N_2 : |S| = c\}$ for some $c \geq 3$.

case	contained type	# games	# games with t types
i	dictator	$CGD(n)$	$CGD(n,t)$
ii	veto	$CGV(n)$	$CGV(n,t)$
iii	passer	$CGP(n)$	$CGP(n,t)$
iv	null	$CGN(n)$	$CGN(n,t)$
v	semi-veto	$CGSV(n)$	$CGSV(n,t)$
vi	semi-passer	$CGSP(n)$	$CGSP(n,t)$

Table 1: Number of complete games with one distinguished type of voters.

PROOF. Lemma 5:

The game has k_1 vetoes. Hence, the vetoes belong to all winning coalitions and therefore they dominate all the other players by the desirability relation, i.e. $i \succsim j$ for all $j \in N$ if i has veto right, and the relation is strict if j does not have veto right. Thus, vetoes belong to the strongest equivalence class, $k_1 = n_1$ and $m_{k,1} = n_1$ for all $1 \leq k \leq r$ in $\mathcal{M}$, otherwise if $m_{k,1} < n_k$ for some k, it would exist a winning coalition $S \subseteq N$ with vector $\overline{m}_k$ and with $i \notin S$ for some veto player i, which would be a contradiction.

The game also have k_2 semi-vetoes. The semi-vetoes belong to the second strongest equivalence class because they are strictly dominated by veto players and they strictly dominate all the remaining players. Indeed, if i is a semi-veto player and j is neither a veto player nor a semi-veto, then $N \setminus \{j\} \notin W$, otherwise $W \subseteq \{S \subseteq N : j \in S\}$, and then j would have veto right, which would be a contradiction. Hence, $N \setminus \{j\} \in W$. Then, $S \cup \{j\} \in W$ implies $S \cup \{i\} \in W$ for all $S \subseteq N \setminus \{i,j\}$, and $T \cup \{j\} \notin W$ and $T \cup \{i\} \in W$ for at least a coalition T (otherwise j would be a semi-veto). Therefore, $k_2 = n_2$, $m_{k,2} = n_2$ for all $1 \leq k \leq r-1$, $m_{r,2} = n_2 - 1$ and $m_{r,i} = n_i$ for all $i = 3, \ldots, t$. The latter condition follows from the fact that for a semi-veto player: $N \setminus \{i\}$ is the unique losing coalition without containing player i. □

3. Enumerations for complete simple games with either the most powerful voters or the least powerful – one distinguished type of voters

In Definition 5 we have exposed six distinguished types of voters. In this section we will consider complete games containing at least one of those six special types of voters. To this end we introduce some notation in Table 1. So $CGV(n)$ e.g. represents the number of complete games with n players and at least one veto player. If the number of different types additionally is restricted we denote the corresponding number by $CGV(n,t)$. If we want to address the respective set of objects instead of their number we use the corresponding curly literals, i.e. $\mathcal{CGD}(n)$ $\mathcal{CGV}(n)$, $\mathcal{CGP}(n)$, $\mathcal{CGN}(n)$ $\mathcal{CGSV}(n)$, $\mathcal{CGSP}(n)$ and $\mathcal{CGD}(n,t)$ $\mathcal{CGV}(n,t)$, $\mathcal{CGP}(n,t)$, $\mathcal{CGN}(n,t)$ $\mathcal{CGSV}(n,t)$, $\mathcal{CGSP}(n,t)$.

The condition that a complete game contains a dictator is very restrictive. Due to our assumption on the ordering of the players, the first player is a dictator and the set of minimal winning coalitions is given by $W^m = \big\{\{1\}\big\}$. All other players then have to be null voters. Thus we have $t = 2$, $n_1 = 1$, $n_2 = n - 1$ and $\mathcal{M} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ unless $n = 1$.

Lemma 8. *$CGD(n) = 1$ for $n \geq 1$ and* $GCD(n,t) = \begin{cases} 1 & \text{if } t = 1, n = 1, \\ 1 & \text{if } t = 2, n \geq 2, \\ 0 & \text{otherwise.} \end{cases}$

The remaining five classes of complete games with one of the distinguished types of voters are pairwise in one–to–one correspondence and therefore their cardinalities coincide.

Theorem 2. *For all positive integers n and t there is a bijection among the sets of complete games*

$$\mathcal{CGV}(n,t),\ \mathcal{CGP}(n,t),\ \mathcal{CGSV}(n,t),\ \mathcal{CGSP}(n,t)$$

and for all positive integers n and $t > 1$ there is a bijection between any of these four classes of games and $\mathcal{CGN}(n,t)$.

Proof. Assume $t = 1$, then $\overline{n} = (n)$ for any complete game with n voters. There is only one complete game with veto $\mathcal{M} = (n)$, only one complete game with passers $\mathcal{M} = (1)$, only one complete game with semi-veto players $\mathcal{M} = (n-1)$, and only one complete game with semi-passers $\mathcal{M} = (2)$.

Thus, the pairwise bijections between pairs of sets $\mathcal{CGV}(n,1)$, $\mathcal{CGP}(n,1)$, $\mathcal{CGSV}(n,1)$ and $\mathcal{CGSP}(n,1)$ are clear.

Assume from now on that $t > 1$. We define a bijection f from $\mathcal{CGV}(n,t)$ to $\mathcal{CGN}(n,t)$; a bijection g from $\mathcal{CGP}(n,t)$ to $\mathcal{CGN}(n,t)$; a bijection h from $\mathcal{CGV}(n,t)$ to $\mathcal{CGSV}(n,t)$ and a bijection k from $\mathcal{CGP}(n,t)$ to $\mathcal{CGSP}(n,t)$. Of course, any other bijection is a composition of some of these bijections or their inverses.

1. *Definition of a bijection f between $\mathcal{CGV}(n,t)$ and $\mathcal{CGN}(n,t)$.*
   Let $(\overline{n}, \mathcal{M}) \in \mathcal{CGV}(n,t)$ be the characteristic invariants of a complete game with t types of voters having a veto. Let $\overline{n} = (n_1, n_2, \ldots, n_t)$ and $m_{i,j}$ be the components of $\mathcal{M}$ as defined in Equation (2) with $1 \leq i \leq r$ and $1 \leq j \leq t$ and $m_{i,1} = n_1$ for all $1 \leq i \leq r$ because the game has veto players.
   In order to define the bijection f, we distinguish two separate cases:

   (i) The game has no null voters.
   Then, f sends vector $(n_1, n_2, n_3, \ldots, n_t)$ to vector $(n_2, n_3, \ldots, n_t, n_1)$ and matrix $\mathcal{M}$ to matrix $\mathcal{M}'$ where for all $1 \leq i \leq r$ and $1 \leq j \leq t-1$ the elements of $\mathcal{M}'$ are defined as $m'_{i,j} = m_{i,j+1}$, whereas $m'_{i,t} = 0$ for all $1 \leq i \leq r$. That is,

$$\mathcal{M}' = \begin{pmatrix} m_{1,2} & \ldots & m_{1,t} & 0 \\ m_{2,2} & \ldots & m_{2,t} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ m_{r,2} & \ldots & m_{r,t} & 0 \end{pmatrix}. \tag{4}$$

   (ii) The game has null voters.
   Then, f is the identity.

   Now we need to check that:
   (a) The map f is well–defined.

(b) The map f is injective.
(c) The map f is surjective.

Well-defined. We have to prove that $f(\overline{n},\mathcal{M}) \in \mathcal{CGN}(n,t)$. This is trivially true if $(\overline{n},\mathcal{M})$ has null voters. Otherwise, let $f(\overline{n},\mathcal{M}) = (\overline{n}',\mathcal{M}')$ where $\mathcal{M}'$ is defined as in Equation (4). Then this pair fulfills the conditions established in Theorem 1-(A), $\overline{n}'$ has t components and $m'_{i,t} = 0$ for all $1 \leq i \leq r$ guarantees the presence of null voters. Hence, $(\overline{n}',\mathcal{M}') \in \mathcal{CGN}(n,t)$.

Injective. Let $(\overline{n},\mathcal{M}),(\overline{m},\mathcal{P}) \in \mathcal{CGV}(n,t)$ with $f(\overline{n},\mathcal{M}) = f(\overline{m},\mathcal{P})$. If $f(\overline{n},\mathcal{M})$ has null voters then $(\overline{n},\mathcal{M}) = f(\overline{n},\mathcal{M}) = f(\overline{m},\mathcal{P}) = (\overline{m},\mathcal{P})$. Otherwise, let $f(\overline{n},\mathcal{M}) = f(\overline{m},\mathcal{P}) = (\overline{h},\mathcal{Q})$, then

$$\overline{h} = (h_1,h_2,\ldots,h_{t-1},h_t) = (n_2,\ldots,n_t,n_1) = (m_2,\ldots,m_t,m_1),$$

thus $n_j = m_j$ for all $1 \leq j \leq t$ and we have $\overline{n} = \overline{m}$; moreover

$$q_{i,j} = m_{i,j+1} = p_{i,j+1} \text{ for all } i, \text{ and for all } j < t.$$

Thus $\mathcal{M}$ and $\mathcal{P}$ have the same dimensions and their components coincide, with the possible exceptions of the components appearing in their respective first columns, but $(\overline{n},\mathcal{M}),(\overline{m},\mathcal{P}) \in \mathcal{CGV}(n,t)$ guarantees that all of them are equal to n_1 and therefore $\mathcal{M} = \mathcal{P}$.
Surjective. Let $(\overline{m},\mathcal{P}) \in \mathcal{CGN}(n,t)$. If it has veto players then $(\overline{m},\mathcal{P}) \in \mathcal{CGV}(n,t)$ and $f(\overline{m},\mathcal{P}) = (\overline{m},\mathcal{P})$. Otherwise, consider $(\overline{n},\mathcal{M})$ defined as follows:

$$n_1 = m_t, \;\; n_i = m_{i-1} \text{ for } 1 < i \leq t,$$

$$m_{i,1} = m_t \text{ for } 1 \leq i \leq r, \; m_{i,j} = p_{i,j-1} \text{ for } 1 < j \leq t.$$

The pair $(\overline{n},\mathcal{M})$ fulfills the conditions of Theorem 1-(A) and has veto players, therefore $(\overline{n},\mathcal{M}) \in \mathcal{CGV}(n,t)$; and $f(\overline{n},\mathcal{M}) = (\overline{m},\mathcal{P})$.

2. *Definition of a bijection g from $\mathcal{CGP}(n,t)$ to $\mathcal{CGN}(n,t)$.*
Let $(\overline{n},\mathcal{M}) \in \mathcal{CGP}(n,t)$ be the characteristic invariants of a complete game with t types having passers. Let $\overline{n} = (n_1,n_2,\ldots,n_t)$ and $m_{i,j}$ be the components of $\mathcal{M}$ as defined in Equation (3) with $1 \leq i \leq r$ and $1 \leq j \leq t$ and $m_{1,1} = 1$, $m_{1,j} = 0$ if $j > 1$ and $m_{i,1} = 0$ if $i > 1$ because the game has passers.
In order to define the bijection g, we distinguish two separate cases:

(i) The game has no null voters.
Then, g sends vector $(n_1,n_2,n_3,\ldots,n_t)$ to vector $(n_2,n_3,\ldots,n_t,n_1)$ and matrix $\mathcal{M}$ to matrix $\mathcal{M}'$ with $r-1$ rows, where for all $1 \leq i \leq r-1$ and $1 \leq j \leq t-1$ the elements of $\mathcal{M}'$ are defined as $m'_{i,j} = m_{i+1,j+1}$, whereas $m'_{i,t} = 0$ for all $1 \leq i \leq r$. That is,

$$\mathcal{M}' = \begin{pmatrix} m_{2,2} & \ldots & m_{2,t} & 0 \\ m_{3,2} & \ldots & m_{3,t} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ m_{r,2} & \ldots & m_{r,t} & 0 \end{pmatrix} \quad (5)$$

(*ii*) The game has null voters.

Then, g is the identity.

Well-defined. All the rows of $\mathcal{M}$ are not pairwise δ-comparable because they fulfill the conditions in Theorem 1-(A), as the game has n_1 vetoes, the first common component n_1 of all the rows has no effect on these comparisons. Thus, the rows of $\mathcal{M}'$ are not pairwise δ-comparable and therefore the pair $(\overline{n}', \mathcal{M}')$ fulfills the conditions in Theorem 1-(A) and g turns the n_1 vetoes into n_1 nulls. Therefore, $(\overline{n}, \mathcal{M}') \in \mathcal{CGN}(n,t)$.
The proof that g is injective and surjective follows the same guidelines as for f.

3. *Definition of a bijection h from $\mathcal{CGV}(n,t)$ to $\mathcal{CGSV}(n,t)$.*
Let $(\overline{n}, \mathcal{M}) \in \mathcal{CGV}(n,t)$ be the characteristic invariants of a complete game with t types of voters having at least one veto player. Let $\overline{n} = (n_1, n_2, \ldots, n_t)$ and $m_{i,j}$ be the components of $\mathcal{M}$ as defined in Equation (2) with $1 \le i \le r$ and $1 \le j \le t$ and $m_{i,1} = n_1$ for all $1 \le i \le r$ because the game has veto players.
In order to define the bijection h, we distinguish two separate cases:

(*i*) The game has no semi-vetoes.

Then, h leaves invariant the vector $(n_1, \ldots, n_t)$ and adds a last row in $\mathcal{M}$ which is $(n_1 - 1, n_2, \ldots, n_t)$ getting $\mathcal{M}'$, i.e. this addition converts the n_1 vetoes into n_1 semi-vetoes since $N \setminus \{i\}$ for all $i \in N_1$ turns into a winning coalition.

(*ii*) The game has semi-vetoes.

Then, h is the identity.

Well-defined. Each row of $\mathcal{M}$, which is also a row of $\mathcal{M}'$, dominates the new last row of $\mathcal{M}'$ because the first component of the former rows are equal to n_1 which is greater than $n_1 - 1$. Moreover, $\sum_{i=1}^{t} m_{k,i} < n - 1$ for all $k = 1, \ldots, r$ since $\sum_{i=1}^{t} m_{k,i} = n - 1$ would imply that coalition $N \setminus \{i\}$ for all $i \in N_1$ would be winning in the original complete game $(\overline{n}, \mathcal{M})$, which is a contradiction with the non-existence of semi-vetoes. Thus, $h(\overline{n}, \mathcal{M}) = (\overline{n}, \mathcal{M}')$ fulfills the conditions of Theorem 1-(A) and h turns the n_1 vetoes into n_1 semi-vetoes. Therefore, $(\overline{n}, \mathcal{M}') \in \mathcal{CGSV}(n,t)$.
The proof that h is injective and surjective follows the same guidelines as for f.

4. *Definition of a bijection k from $\mathcal{CGP}(n,t)$ to $\mathcal{CGSP}(n,t)$*
Let $(\overline{n}, \mathcal{M}) \in \mathcal{CGP}(n,t)$ be the characteristic invariants of a complete game with passers and with t types of voters. Let $\overline{n} = (n_1, \ldots, n_t)$ and $m_{i,j}$ be the components of $\mathcal{M}$ as defined in Equation (3) with $1 \le i \le r$ and $1 \le j \le t$.
In order to define the bijection k, we distinguish two separate cases:

(*i*) The game has no semi-passers.

Then, k leaves invariant vector $(n_1, \ldots, n_t)$ and transform the first row in $\mathcal{M}$, which is $(1, 0, \ldots, 0, 0)$, into $(1, 0, \ldots, 0, 1)$, while the rest of rows keep invariant to get $\mathcal{M}'$.

(*ii*) The game has semi-passers.
Then, k is the identity.

Well-defined. Of course the first new row of $\mathcal{M}'$ dominates the other rows because $(1,0,\ldots,0,1)$ δ-dominates $(1,0,\ldots,0,0)$ which δ-dominates the rest of the rows of $\mathcal{M}'$. Conversely, as the game has no semi-passers it does not have any row of type $(0,1,0,\ldots,0,1)$ and all of its rows δ-dominate $(0,1,0,\ldots,0,1)$ and as $m_{i,1} = 0$ for all $i > 1$ it follows that each row $k = 2,\ldots r$ of $\mathcal{M}$ does not δ-dominate $(1,0,\ldots,0,1)$, thus they are pairwise non-δ-comparable. Thus, $k(\overline{n},\mathcal{M}) = (\overline{n},\mathcal{M}')$ fulfills the conditions of Theorem 1-(A) and k turns the n_1 passers into n_1 semi-passers. Therefore, $(\overline{n},\mathcal{M}') \in \mathcal{CGSP}(n,t)$.
The proof that k is injective and surjective follows the same guidelines as for f.

□

We remark that we had to exclude the set $\mathcal{CGN}(n)$ from the bijections in Theorem 2 for $t = 1$ since there are no simple games with only null voters due to the definition of a simple game.

Corollary 1. *For all positive integers n and t we have*

$$CGV(n,t) = CGP(n,t) = CGSV(n,t) = CGSP(n,t).$$

For $t > 1$ we additionally have $CGN(n,t) = CGV(n,t)$ and $CGN(n,1) = CGV(n,1) - 1 = 0$.

Having the enumeration results for complete games with n voters and at most 2 types of voters at hand, we can conclude enumeration formulas for complete games with $t < 4$ types of voters, where at least one of these equivalence classes corresponds to a distinguished type of voters. We state the results for $CGV(n,t)$ only. Combining them with Corollary 1 yields five simultaneous equivalent enumerations.

Proposition 1.

1. $CGV(n,1) = 1$,
2. $CGV(n,2) = \dfrac{n(n-1)}{2}$ *(whenever $n \geq 2$),*
3. $CGV(n,3) = F(n+7) - \dfrac{1}{2}(n^3 + 2n^2 + 13n + 26)$ *(whenever $n \geq 4$),*
   where $F(n)$ are the Fibonacci numbers which form a sequence of integer numbers defined by the following recurrence relation: $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for all $n > 1$.

Proof.

1. If $t = 1$, we have $\overline{n} = \mathcal{M} = (n)$. Therefore $CGV(n,1) = 1$.

2. If $t = 2$, let $\overline{n} = (n_1, n_2)$ for a given game. As the game must have veto players all the entries in the first column must be n_1. As the rows of $\mathcal{M}$ must be non-comparable by the δ-relation, $\mathcal{M}$ only should have a row with the only requirement that $m_{1,2}$ must be different from n_2, otherwise all players would be equivalent and therefore $t = 1$, which would be a contradiction. Thus we have

$$CGV(n,2) = \sum_{n_1=1}^{n-1} \sum_{m_{1,2}=0}^{\overbrace{n-n_1-1}^{=n_2-1}} 1 = \frac{n(n-1)}{2}.$$

3. If we remove the null voters from a complete game with t types of voters (having at least one null voter), we obtain a complete game with $t-1$ types of voters without null voters. We remark that due to the definition of a simple game it cannot consist of null voters only. Thus we have

$$CGV(n,3) = \sum_{i=1}^{n} CG(n-i,2) - CGV(n-i,2) = \sum_{i=1}^{n-2} CG(n-i,2) - CGV(n-i,2).$$

Inserting $CGV(n,2) = \dfrac{n(n-1)}{2}$ and $CG(n,2) = F(n+6) - (n^2+4n+8)$, see Equation (1), yields

$$\begin{aligned} CGV(n,3) &= \sum_{k=2}^{n-1} \left[F(k+6) - \frac{k(k-1)}{2} - (k^2+4k+8) \right] \\ &= F(n+7) - \frac{1}{2}(n^3+2n^2+13n+26) \end{aligned}$$

□

Starting with $n = 4$ the first ten numbers of the sequence $CGV(n,3)$ are

$$2, 11, 37, 98, 225, 470, 919, 1713, 3082, 5400.$$

Asymptotically we have

$$\lim_{n\to\infty} \frac{CGV(n,3)}{CG(n,2)} = \lim_{n\to\infty} \frac{F(n+7)+O(n^3)}{F(n+6)+O(n^2)} = \lim_{n\to\infty} \frac{F(n+7)}{F(n+6)} = \frac{1+\sqrt{5}}{2}.$$

Thus, when n is fixed and large enough, the number of complete games with three types of voters having veto players (or semi-veto, passer, semi-passer or null voters) is almost equal to the number of complete games with only two types of voters multiplied by $\dfrac{1+\sqrt{5}}{2}$.

We remark that the bijections from Theorem 2 respect the property of being weighted, which is not too hard to prove but we omit this part because it is not needed for the purposes of this paper. Moreover, the bijections keep games within the class of so-called

α-roughly weighted games, where the fraction of the weight of the heaviest losing coalition divided by the weight of lightest winning coalition is at most $\alpha \geq 1$, see [20], where the authors have introduced this as one of three hierarchies in order to classify simple games. The special case $\alpha = 1$ corresponds to the better known class of roughly-weighted games, see e.g. [21].

4. Enumerations with two distinguished type of voters in a complete simple game

Continuing the considerations from Section 3 we study complete games containing at least two of the six distinguished types from Definition 5. As mentioned in the beginning of Section 3 each complete game with a dictator contains $n-1$ null voters. By $CGDN(n)$ we denote the number of complete games with n voters containing a dictator and at least one null and by $CGDN(n,t)$ we denote the number of these objects additionally restricted to exactly t types of voters. From Lemma 8 we conclude:

Lemma 9. *For* $n \geq 2$ *we have* $\text{CGDN}(n) = \text{CGD}(n) = 1$ *and*

$$\text{CGDN}(n,t) = \text{CGD}(n,t) = \begin{cases} 1 & \textit{if } \; t = 2, \\ 0 & \textit{otherwise.} \end{cases}$$

For $n \leq 1$ *all four counts are zero.*

Each dictator has also a veto and is a passer. So for each subset $\mathcal{S} \subseteq \{V, P\}$ we have $CGD\mathcal{S}N(n) = CGD\mathcal{S}(n)$ and $CGD\mathcal{S}N(n,t) = CGD\mathcal{S}(n,t)$, extending the previously used notation for the number of complete games with presence of some distinguished types of voters in a natural way.

If a complete game containing a dictator contains a semi-veto or a semi-passer then we have $n = 2$ and $t = 2$.

Lemma 10. *For two subsets* $\mathcal{S}_1 \subseteq \{V, P, N\}$ *and* $\mathcal{S}_2 \subseteq \{SV, SP\}$ *with* $|\mathcal{S}_2| \geq 1$ *we have*

$$\text{CGD}\mathcal{S}_1\mathcal{S}_2(n) = \begin{cases} 1 & \textit{if } \; n = 2, \\ 0 & \textit{otherwise} \end{cases} \quad \textit{and} \quad \text{CGD}\mathcal{S}_1\mathcal{S}_2(n,t) = \begin{cases} 1 & \textit{if } \; n = t = 2, \\ 0 & \textit{otherwise.} \end{cases}$$

If a complete game contains a passer and a veto then these roles must be taken by the same player, which then is a unique dictator, and for each subset $\mathcal{S} \subseteq \{SV, SP, N\}$ we have $CGVP\mathcal{S}(n) = CGD\mathcal{S}(n)$ and $CGVP\mathcal{S}(n,t) = CGD\mathcal{S}(n,t)$, compare the enumeration formulas in Lemma 8, Lemma 9, and Lemma 10.

Next we consider the simultaneous occurrence of a semi-veto and a semi-passer in a complete game. By $CGSVSP(n)$ we denote the number of complete games with n voters containing at least one semi-veto and at least one semi-passer. By $CGSVSP(n,t)$ we denote the number of these objects additionally restricted to exactly t types of voters. Here, and in the following definitions, we permit the somewhat artificial situation that the two types of distinguished voters could be taken by the same player.

Lemma 11. *For* $n \geq 1$ *we have*

$$\text{CGSVSP}(n) = \begin{cases} 0 & \textit{if } \; n = 1, \\ 2 & \textit{if } \; n = 3, \\ 1 & \textit{otherwise} \end{cases}$$

and

$$\mathrm{CGSVSP}(n,t) = \begin{cases} 1 & \textit{if } \; t=1, n=3, \\ 1 & \textit{if } \; t=2, n\ge 2, \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. We assume that voter 1 is a semi-passer. From the definition of a semi-passer we conclude $\{1\} \notin W$, $\{1,2\},\{1,3\},\dots,\{1,n\} \in W$. For $n \ge 4$ voter 1 is the unique voter being contained in each winning coalition besides $N\backslash\{i\}$. Thus voter 1 also has to be the unique semi-veto and we conclude

$$W^m = \big\{\{1,2\},\{1,3\},\dots,\{1,n\},\{2,3,\dots,n\}\big\}.$$

Since the simple game given by W^m has the weighted representation $[n{-}1; n{-}2; \overbrace{1,1,\dots,1}^{n-1}]$ it is complete. For $n=1$ no simple game containing a semi-veto and a semi-passer exists. If $n=2$ and voter 1 is a semi-passer, then we have $\{1\} \notin W$ and $\{1,2\} \in W$. Since for each semi-veto player i the coalition $N\backslash\{i\}$ is winning, we conclude that voter 1 is also a semi-veto player and $\{2\}$ is a winning coalition, so that we have $W^m = \big\{\{2\}\big\}$. Here player 2 is a dictator, in contrast to our usual ordering of the players. We remark that player 1 is also a null voter and thus has three roles in this case. For $n=3$ we have the simple games uniquely characterized by $W^m = \big\{\{1,2\},\{1,3\},\{2,3\}\big\}$ and $W^m = \big\{\{1,2\},\{1,3\}\big\}$. In the first case voter 1 is both the unique semi-passer and the unique semi-veto. In the second case all three players are semi-passers and semi-vetoes. In the third case voter 1 is the unique semi-passer and players 2 and 3 are the semi-vetoes. □

We remark that all simple games with at most three voters are complete (and weighted).

Next we consider complete games containing vetoes and semi-passers. The corresponding counts are denoted by $CGVSP(n)$ and $CGVSP(n,t)$.

Lemma 12. *For $n \ge 1$ we have*

$$\mathrm{CGVSP}(n) = \begin{cases} 0 & \textit{if } \; n=1, \\ 2 & \textit{if } \; n=2, \\ 1 & \textit{otherwise} \end{cases} \quad \textit{and} \quad \mathrm{CGVSP}(n,t) = \begin{cases} 1 & \textit{if } \; t=1,\, n=2, \\ 1 & \textit{if } \; t=2,\, n\ge 2, \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. For $n=1$ no semi-passer is possible and for $n=2$ there are only three complete games. So we can easily check that the complete simple game given by $W^m = \big\{\{1\},\{2\}\big\}$ does contain neither a veto player nor a semi-passer. In the complete game given by $W^m = \big\{\{1\}\big\}$ voter 1 has a veto and voter 2 is a semi-passer, so that we have $t=2$. In the complete game given by $W^m = \big\{\{1,2\}\big\}$ both players are semi-passers and vetoes, so that we have $t=1$.

For $n \ge 3$ we assume that voter 1 is a semi-passer so that $\{1,2\},\{1,3\},\dots,\{1,n\} \in W$ and $\{1\} \notin W$. The only possible veto player is voter 1 and no further vetoes or semi-passers can be present. Thus voter 1 forms its own equivalence class of voters N_1. Since player 1 has a veto we conclude $W^m = \big\{\{1,2\},\{1,3\},\dots,\{1,n\}\big\}$ and $t=2$. □

Similarly we denote by $CGPSV(n)$ and $CGPSV(n,t)$ the counts for complete games containing passers and semi-vetoes.

Lemma 13. *For $n \geq 1$ we have*

$$\mathrm{CGPSV}(n) = \begin{cases} 0 & \text{if } n = 1, \\ 2 & \text{if } n = 2, \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathrm{CGPSV}(n,t) = \begin{cases} 1 & \text{if } t = 1, n = 2, \\ 1 & \text{if } t = 2, n \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. For $n = 1$ no semi-veto is possible and for $n = 2$ there are only three complete games. Assuming w.l.o.g. that voter 1 is a passer the remaining possibilities are given by $W^m = \big\{\{1\}, \{2\}\big\}$ and $W^m = \big\{\{1\}\big\}$. In the first mentioned complete game both players are passers and semi-vetoes so that we have $t = 1$. In the second case voter 1 is a passer and voter 2 a semi-veto so that we have $t = 2$.

Due to the definition of a semi-veto for $n \geq 3$ the only possible semi-veto is player 1 and there can be no other passers besides player 1. Thus we have $W^m = \big\{\{1\}, \{2, \ldots, n\}\big\}$. □

Even more mapping to dual games is a bijection between the corresponding classes of complete games of Lemma 12 and Lemma 13, since the pairs vetoes/passers and semi-vetoes/semi-passers are interchanged, see Definition 6 and the comment thereafter.

Due to the definition of a semi-passer (for $n \geq 2$) a complete game cannot contain both a semi-passer and a null voter. Similarly a complete game cannot contain both a semi-veto and a null voter.

The remaining pairs of distinguished types of voters are veto/semi-veto, passer/semi-passer, veto/null, and passer/null. The respective counts are denoted by the terms $CGVSV(n)$, $CGPSP(n)$, $CGVN(n)$, $CGPN(n)$, $CGVSV(n,t)$, $CGPSP(n,t)$, $CGVN(n,t)$, and $CGPN(n,t)$. We denote the corresponding classes by $\mathcal{CGVSV}(n)$, $\mathcal{CGPSP}(n)$, $\mathcal{CGVN}(n)$, $\mathcal{CGPN}(n)$, $\mathcal{CGVSV}(n,t)$, $\mathcal{CGPSP}(n,t)$, $\mathcal{CGVN}(n,t)$, and $\mathcal{CGPN}(n,t)$.

Theorem 3. *For all positive integers $n > 1$ and $t > 1$ there is a bijection between $\mathcal{CGVSV}(n,t)$, $\mathcal{CGPSP}(n,t)$, $\mathcal{CGVN}(n,t)$ and $\mathcal{CGPN}(n,t)$.*

PROOF. We first define a bijection h between $\mathcal{CGVN}(n,t)$ and $\mathcal{CGPN}(n,t)$. If $(\overline{n}, \mathcal{M}) \in \mathcal{CGVN}(n,t)$ then let

$$h(\overline{n}, \mathcal{M}) = (\overline{n}, \mathcal{M}^*)$$

where $(\overline{n}, \mathcal{M}^*)$ is the dual game of $(\overline{n}, \mathcal{M})$.

Now we need to check that:

1. The map h is well–defined.
2. The map h is injective.
3. The map h is surjective.

Well-defined. We have to prove that $h(\overline{n}, \mathcal{M}) = (\overline{n}, \mathcal{M}^*) \in \mathcal{CGPN}(n,t)$. This is trivially true because the dual of a complete game is a complete game too, $i \in N$ is null in (N, W) *if and only if* $i \in N$ is null in (N, W^*), and $i \in N$ has veto in (N, W) *if and only if* $i \in N$ is a passer in (N, W^*).

Injective. Let $(\overline{n}, \mathcal{M}), (\overline{m}, \mathcal{P}) \in \mathcal{CGVN}(n,t)$ with $h(\overline{n}, \mathcal{M}) = h(\overline{m}, \mathcal{P})$. Let $h(\overline{n}, \mathcal{M}) = h(\overline{m}, \mathcal{P}) = (\overline{g}, \mathcal{Q})$. Because of the definition of h we have

$$\overline{g} = \overline{n} = \overline{m} \qquad \text{and} \qquad \mathcal{Q} = \mathcal{M}^* = \mathcal{P}^*$$

and after applying duality to the last equation we obtain the desired equality, i.e. $\mathcal{Q}^* = (\mathcal{M}^*)^* = \mathcal{M}$ and $\mathcal{Q}^* = (\mathcal{P}^*)^* = \mathcal{P}$
Surjective. Let $(\overline{m}, \mathcal{P}) \in \mathcal{CGPN}(n,t)$, then it is clear that $(\overline{m}, \mathcal{P}^*) \in \mathcal{CGVN}(n,t)$ and $h(\overline{m}, \mathcal{P}^*) = (\overline{m}, \mathcal{P})$.

Next we remark that the same mapping is also a bijection from $\mathcal{CGVSV}(n,t)$ to $\mathcal{CGPSP}(n,t)$. As before all three conditions can be easily checked.

Finally we define a bijection h' between $\mathcal{CGVSV}(n,t)$ and $\mathcal{CGVN}(n,t)$. Let $(\overline{n}, \mathcal{M}) \in \mathcal{CGVSV}(n,t)$, with $\overline{n} = (n_1, \ldots, n_t)$ and $m_{i,j}$ be the components of $\mathcal{M}$ as defined in Lemma 5-(v) with $1 \le i \le r$ and $1 \le j \le t$. With this we set $h'(\overline{n}, \mathcal{M}) = (\overline{n}', \mathcal{M}')$, where

$$\begin{aligned} \overline{n}' &= (n_1, n_3, \ldots, n_t, n_2), \\ \mathcal{M}' &= \begin{pmatrix} m_{1,1} & m_{1,3} & \cdots & m_{1,t} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{r-1,1} & m_{r-1,3} & \cdots & m_{r-1,t} & 0 \end{pmatrix}. \end{aligned}$$

We note that we have $r \ge 2$. In other words, we have shifted the second equivalence class of voters to the tth equivalence class, while replacing the second column $(m_{1,2}, \ldots, m_{r,2})^T$ by the all-zero vector and deleted the last row of $\mathcal{M}$.

Due to Lemma 5-(v) the second column and the last row of $\mathcal{M}$ are uniquely characterized by the values of $n_1, \ldots, n_t$. So it remains to check that the resulting games are complete games with t equivalence classes of voters, i.e. that they satisfy the conditions from Theorem 1, which can be done easily. □

Corollary 2. *For all positive integers $n > 1$ and $t > 1$ we have*

$$\mathrm{CGVSV}(n,t) = \mathrm{CGPSP}(n,t) = \mathrm{CGVN}(n,t) = \mathrm{CGPN}(n,t).$$

Proposition 2.

1. $\mathrm{CGVN}(n,2) = n - 1$ *(whenever $n \ge 2$),*
2. $\mathrm{CGVN}(n,3) = \dfrac{(n-1)(n-2)(n-3)}{6}$ *(whenever $n \ge 4$),*
3. $\mathrm{CGVN}(n,4) = F(n+8) - \dfrac{1}{6}(n^4 - 2n^3 + 26n^2 + 47n + 132)$ *(whenever $n \ge 5$).*

Proof.

1. Assume $t = 2$, for each vector (n_1, n_2) with $n_1 + n_2 = n$ with $0 < n_1 < n$ there is a unique matrix which is $\mathcal{M} = (n_1, 0)$.
2. Assume $t = 3$, for each vector (n_1, n_2, n_3) there are $n_2 - 1$ matrices of type $\mathcal{M} = (n_1, a, 0)$ where $0 < a < n_2$. Hence,

$$CGVN(n,3) = 1(n-3) + 2(n-4) + \cdots + (n-3)1 = \sum_{i=1}^{n-1} i \cdot (n-2-i)$$

which coincides with the given expression since $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$ and $\sum_{i=1}^{n-1} i^2 = \frac{n(n-1)(2n-1)}{6}$.

3. If we remove the null voters from a complete game with t types of voters (having at least one null voter), we obtain a complete game with $t-1$ types of voters without null voters. Thus we have the recurrence relation

$$\begin{aligned} CGVN(n,4) &= \sum_{i=1}^{n} CGV(n-i,3) - CGVN(n-i,3) \\ &= \sum_{i=1}^{n-4} CGV(n-i,3) - CGVN(n-i,3) \end{aligned}$$

and substituting in it the two known results

$$CGV(n,3) = F(n+7) - \frac{1}{2}(n^3 + 2n^2 + 13n + 26)$$

$$CGVN(n,3) = \frac{(n-1)(n-2)(n-3)}{6}$$

yields (for $n \geq 5$)

$$\begin{aligned} CGVN(n,4) &= \sum_{k=4}^{n-1} \left[F(k+7) - \frac{1}{2}(k^3 + 2k^2 + 13k + 26) - \frac{1}{6}(k-1)(k-2)(k-3) \right] \\ &= F(n+8) - \frac{1}{6}(n^4 - 2n^3 + 26n^2 + 47n + 132) \end{aligned}$$

□

Starting with $n = 5$ the first ten numbers of the sequence $CGVN(n,4)$ are

$$1, 8, 35, 113, 303, 717, 1552, 3145, 6062, 11242.$$

Asymptotically we have

$$\lim_{n\to\infty} \frac{CGVN(n,4)}{CG(n,2)} = \lim_{n\to\infty} \frac{F(n+8) + O(n^4)}{F(n+6) + O(n^2)} = \lim_{n\to\infty} \frac{F(n+8)}{F(n+6)} = \left(\frac{1+\sqrt{5}}{2} \right)^2$$

Thus, when n is fix and large enough, the number of complete games with four types of voters having vetoes and nulls (or one of the combinations passers/nulls, vetoes/semi-vetoes, or passers/semi-passer) is almost equal to the number of complete games with two types of voters multiplied by $\left(1+\sqrt{5}\right)^2/4$.

5. Enumerations with more than two distinguished type of voters in a complete simple game

Continuing the considerations from the previous two sections we study complete games containing at least three of the six distinguished types from Definition 5. Complete simple games containing a dictator and at least another distinguished type of voters are completely treated at the beginning of Section 4. So in the following we assume that no dictator is present, including the combination of a passer and a veto, which then would be a dictator. Summarizing the results from Section 4 we state that only the following seven combinations of two distinguished type of voters are possible:

- semi-passer and semi-veto
- veto and semi-veto
- passer and semi-passer
- passer and semi-veto
- veto and semi-passer
- veto and null voter
- passer and null voter

If a complete game contains at least one semi-passer and at least one semi-veto then no other distinguished types of voters can occur. If we represent the possible combinations of two distinguished types of voters by an edge, we obtain a quadrangle on the set $\{\text{veto}, \text{passer}, \text{semi-veto}, \text{semi-passer}\}$ of vertices. Thus there are no complete games with at least three distinguished types of voters.

6. Conclusion and future research

In this paper we have studied complete games containing at least one voter of a list of distinguished types of voters and provided several bijections between the corresponding classes of voting systems. This contributes to the program of numerical characterization and classification of voting systems initiated by von Neumann and Morgenstern [32].

It turned out that all of the counts $CGV(n,3)$, $CGN(n,3)$, $CGP(n,3)$, $CGSV(n,3)$, $CGSP(n,3)$, $CGVN(n,4)$, $CGPN(n,4)$, $CGPSP(n,4)$, and $CGVSV(n,4)$ of complete games with distinguished and two additional types of voters belong to the class $\Theta\left(\left((1+\sqrt{5})/2\right)^n\right)$ as well as $CG(n,2)$. Thus, the addition of just one of the following set of voters: vetoers, passers, nulls, vetoers and nulls, passers or nulls, semi-vetoers or semi-passers in a complete game with two types of voters does not alter the asymptotic behavior. For a fix n the number of these games is equal to $k \cdot F(n+6) + P(n)$ where for each case k is a positive constant that takes one of the two values $(1+\sqrt{5})/2$ or $\left(1+\sqrt{5}\right)^2/4$, and P is a polynomial.

The exact enumeration formulas are mainly based on the previously determined enumeration of complete games with two types of voters, i.e. $CG(n,2)$. A quite natural next step now is to consider complete games with three types of voters. So far we were only able to compute the first few exact values. For $4 \le n \le 21$ the sequence $CG(n,3)$ is given by

$$6, 50, 262, 1114, 4278, 15769, 58147, 221089, 886411, 3806475, 17681979,$$
$$89337562, 492188528, 2959459154, 19424078142, 139141985438,$$
$$1087614361775, 9274721292503.$$

For $t \ge 4$ and $n \ge 10$ we could only compute the additional values $CG(10,4) = 4570902$, $CG(11,4) = 59776637$, $CG(12,4) = 1047858496$, $CG(13,4) = 26000281487$, $CG(10,5) = 412734188$, $CG(11,5) = 29086472429$, and $CG(10,6) = 42427707348$.

Significant sub-classes of the studied structures are those which are weighted games and, or more generally, roughly weighted games being complete. Even though that there exist combinatorial characterizations of weighted games (see [13] and [40]) and of roughly weighted games (see [21]), very little is known on closed enumeration formulas for these much more restrictive voting structures, besides May's Theorem for voting systems with only one type of voters.

Using the characterization of weighted games with two types of voters given in [14] we can at least compute some exact values of $WG(n,2)$ without generating the entire class of corresponding complete games with two types of voters. We would like to remark that it took only 19 days of computation time to compute the values of $WG(n,2)$ for all $n \le 200$. Two examples are given by $WG(100,2) = 27970501$ and $WG(200,2) = 851946591$. Having these numerical data at hand we observe that $WG(n,2) \approx 0.002531n^5 + O(n^4)$. And indeed it is not too hard to come up with an upper bound of $WG(n,2) \le \frac{n^5}{15} + 4n^4$, see [11], and a similar lower bound. The determination of an exact enumeration formula for $WG(n,2)$ seems to be an interesting but resolvable research problem.

Acknowledgments

The authors are grateful to the two referees of this paper for their interesting comments that contributed to improve the original submitted version.

References

References

[1] J.M. Alonso-Meijide, J.M. Bilbao, B. Casas-Méndez, and J.R. Fernández. Weighted multiple majority games with unions: Generating functions and applications to the European Union. *Eur. J. Oper. Res.*, 198:540–554, 2009.

[2] H. Aziz. *Algorithmic and complexity aspects of simple coalitional games*. PhD thesis, University of Warwick, 2009.

[3] R. Berghammer, S. Bolus, A. Rusinowska, and H. de Swart. A relation-algebraic approach to simple games. *Eur. J. Oper. Res.*, 210:68–80, 2011.

[4] S. Bolus. Power indices of simple games and vector-weighted majority games by means of binary decision diagrams. *Eur. J. Oper. Res.*, 210:258–272, 2011.

[5] E. Boros, P.L. Hammer, T. Ibaraki, and K. Kawakawi. Identifying 2-monotonic positive boolean functions in polynomial time, in ISA'91 Algorithms, W.L. Hsu and R.C.T. Lee eds. *Springer Lecture Notes in Computer Sciences (LNCS)*, 557:104–115, 1991.

[6] E. Boros, P.L. Hammer, T. Ibaraki, and K. Kawakawi. Polynomial time recognition of 2-monotonic positive functions given by an oracle. *SIAM Journal of Computing*, 26:93–109, 1997.

[7] F. Carreras and J. Freixas. Complete simple games. *Math. Social Sci.*, 32:139–155, 1996.

[8] W.D. Cook. Distance-based and ad hoc consensus models in ordinal preference ranking. *Eur. J. Oper. Res.*, 172:369–385, 2006.

[9] D.S. Felsenthal and M. Machover. *The measurement of voting power*. Edward Elgar Publishing Limited, Cheltenham, UK, 1998.

[10] J. Freixas. On ordinal equivalence of the Shapley and Banzhaf values for cooperative games. *Internat. J. Game Theory*, 39:513–527, 2010.

[11] J. Freixas and S. Kurz. On minimal integer representations of weighted games. submitted, 2011 (http://arxiv.org/abs/1103.0868).

[12] J. Freixas, D. Marciniak, and M. Pons. On the ordinal equivalence of the Johnston, Banzhaf and Shapley power indices. *Eur. J. Oper. Res.*, 216:367–375, 2012.

[13] J. Freixas and X. Molinero. Simple games and weighted games: a theoretical and computational viewpoint. *Discrete Appl. Math.*, 157:1496–1508, 2009.

[14] J. Freixas, X. Molinero, and S. Roura. Complete voting systems with two types of voters: weightedness and counting. *Ann. Oper. Res.*, 193:273–287, 2012.

[15] J. Freixas and M.A. Puente. Dimension of complete simple games with minimum. *Eur. J. Oper. Res.*, 188:555–568, 2008.

[16] J. Freixas and M.A. Puente. Complete games with minimum. *Ann. Oper. Res.*, 84:97–109, 1998.

[17] J. Freixas and M.A. Puente. Reliability importance measures of the components in a system based on semivalues and probabilistic values. *Ann. Oper. Res.*, 109:331–342, 2002.

[18] J. Freixas and W.S. Zwicker. Weighted voting, abstention, and multiple levels of approval. *Soc. Choice Welf.*, 21:399–431, 2003.

[19] T. Gvozdeva, A. Hameed, and A. Slinko. Weightedness and structural characterization of hierarchical simple games. *Math. Social Sci. (to appear)*, 2012.

[20] T. Gvozdeva, L.A. Hemaspaandra, and A. Slinko. Three hierarchies of simple games parameterized by “resource“ parameters. *Int. J. Game Theory*, to appear, doi: 10.1007/s00182-011-0308-4.

[21] T. Gvozdeva and A. Slinko. Weighted and roughly weighted simple games. *Math. Social Sci.*, 61:20–30, 2011.

[22] S.T. Hu. *Threshold Logic*. Univ. of California Press, Berkeley and Los Angeles, USA, 1965.

[23] J.R. Isbell. A class of simple games. *Duke Math. J.*, 25:423–439, 1958.

[24] K. Jabeur, J. Martel, and S. Khelifa. A distance-based collective preorder integrating the relative importance of the group's members. *Group Decision and Negotiation*, 13:327–349, 2004.

[25] D.M. Kilgour. A formal analysis of the amending formula of Canada's Constitution Act. *Canadian Journal of Political Science*, 16:771–777, 1983.

[26] S. Kurz. On minimum sum representations for weighted voting games. *Annals of Operations Research*, 196(1):361–369, 2012

[27] S. Kurz and N. Tautenhahn. On Dedekind's problem for complete simple games. to appear in *Internat. J. Game Theory* (http://arxiv.org/abs/1001.3045).

[28] L.D. Lambo and J. Moulen. Ordinal equivalence of power notions in voting games. *Theory and Decision*, 53:313–325, 2002.

[29] M. Le Breton, M. Montero, and V. Zaporozhets. Voting power in the EU council of ministers and fair decision making in distributive politics. *Math. Social Sci.*, 63:159–173, 2012.

[30] D. Leech. Voting power in the governance of the International Monetary fund. *Ann. Oper. Res.*, 109:375–397, 2002.

[31] K.O. May. A set of independent, necessary and sufficient conditions for simple majority decision. *Econometrica*, 20:680–684, 1952

[32] J. von Neumann and O. Morgenstern. *Theory of games and economic behavior.* Princeton University Press, New Jersey, USA, 1944.

[33] W. Obata and H. Ishii. A method for discriminating efficient candidates with ranked voting data. *Eur. J. Oper. Res.*, 151:233–237, 2003.

[34] B. Peleg. Coalition formation in simple games with dominant players. *Internat. J. Game Theory*, 10:11–33, 1981.

[35] G.J. Simmons. How to (really) share a secret. In *Proceedings of the 8th Annual International Cryptology Conference on Advances in Cryptology*, pages 390–448. Springer-Verlag, London, UK, 1990.

[36] D.R. Stinson. An explanation of secret sharing schemes. *Des. Codes Cryptography*, 2:357–390, 1992.

[37] T. Sueyoshi, J. Shang, and W.H.Chiang. A decision support framework for internal audit priorization in a rental car company: A combined use between DEA and AHP. *Eur. J. Oper. Res.*, 199:219–231, 2009.

[38] T. Tassa. Hierarchical threshold secret sharing. *J. Cryptology*, 20:237–264, 2007.

[39] A.D. Taylor and A. Pacelli, *Mathematics and Politics, second edition.* Springer Verlag, New York, USA, 2008.

[40] A.D. Taylor and W.S. Zwicker. A characterization of weighted voting. *Proceedings of the American Mathematical Society*, 115:1089–1094, 1992.

[41] A.D. Taylor and W.S. Zwicker. *Simple games: desirability relations, trading, and pseudoweightings.* Princeton University Press, New Jersey, USA, 1999.