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Integral point sets over $\mathbb{Z}_n^m$

An integral point set $\mathcal{P}$ is a set of n points in the m-dimensional space $\mathbb{K}^m$ with pairwise integral distances. The largest occurring distance is called its diameter. From the combinatorial point of view one is interested in the minimum possible diameter $d(n, m)$ for given n and m. Most known results are for $\mathbb{K} = \mathbb{R}$ with the Euclidean metric [2, 3], $\mathbb{K} = \mathbb{Z}$ [4], and some for Banach spaces [1].

To gain some insight in the cases $\mathbb{K} = \mathbb{R}, \mathbb{Z}$ we consider the homomorphism $x \mapsto x + \mathbb{Z}n$, which leads us to integral point sets for $\mathbb{K} = \mathbb{Z}\backslash\mathbb{Z}n$ and $\mathbb{K} = \mathbb{R}\backslash\mathbb{Z}n$. By $\mathcal{I}(n, m)$ we denote the maximum number of points in $\mathbb{Z}_n^m$ with pairwise integral distances. Some properties are:

Theorem. For an odd integer n we have $\mathcal{I}(2n, m) = 2^m \cdot \mathcal{I}(n, m)$.

Integral point sets over $\mathbb{Z}_3^m$ correspond to subsets of $\mathbb{Z}_3^m$ with Hamming distance $h(u, v) \not\equiv 2 \mod 3$ for all points u, v.

Lemma. For two coprime integers a and b we have $\mathcal{I}(a \cdot b, m) \geq \mathcal{I}(a, m) \cdot \mathcal{I}(b, m)$.

$m\backslash n$	3	4	5	6	7	8	9	10	11	12
1	3	4	5	6	7	8	9	10	11	12
2	3	8	5	12	7	16	27	20	11	24
3	4	16	25	32	8	64	81	200	11	64
4	9	32	25	144	49	512	324	400	121	288
5	27	128	125	864	343	2048	≥ 893	4000	≥ 1331	≥ 3456
6	33	256	≥ 125	2112		≥ 15296		≥ 8000		≥ 8448
7	≥ 35	1024		≥ 4480		≥ 81792				≥ 35840

Table 1: Values of $\mathcal{I}(n, m)$ for small parameters n and m.

References

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