CLASSES OF COMPLETE SIMPLE GAMES THAT ARE ALL WEIGHTED

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ABSTRACT. Important decisions are likely made by groups of agents. Thus group decision making is very common in practice. Very transparent group aggregating rules are given by weighted voting, where each agent is assigned a weight. Here a proposal is accepted if the sum of the weights of the supporting agents meets or exceeds a given quota. We study a more general class of binary voting systems – complete simple games – and propose an algorithm to determine which sub classes, parameterized by the agent's type composition, are weighted.

Keywords: Complete Simple Games, Weighted Games, Voting, Group Decision Making.

1. INTRODUCTION

Weighted voting is a method for group decision making. For simplicity, we assume that for each proposal on a certain issue the group members, called agents for brevity, options are either to vote "yes" or "no". The aggregated group decision then is also either "yes" or "no". Those procedures are called binary voting systems or games in the literature. The special case of a weighted game consists of a quota $q > 0$ and weights $w_i \geq 0$ for every participating agent. With this, the aggregated group decision is "yes" if and only if the summed weights of the supporters of a given proposal meets or exceeds the quota. For n agents, such a game is denoted by $[q; w_1, \ldots, w_n]$.

Weighted voting systems are commonly applied whenever not all agents are considered to be equal. Reasons may lie in heterogeneous competencies for different issues, see e.g. (Grofman et al., 1983). In stock corporations, weights can arise as the number of shares that each shareholder owns, see e.g. (Leech, 2013). In two-tier voting systems like the US Electoral College or the EU Council of Ministers, agents vote as a block or represent countries with different population sizes, which then have to be mapped to appropriate weights, see e.g. (Maaser and Napel, 2007).

Weighted games form a very concrete, compact, and well-studied sub class of group decision rules. Nevertheless, some practical decision rules of legislative bodies do not correspond to weighted games, like e.g. the present rule of the EU Council of Ministers. In Section 2, we introduce and motivate the important super class of complete simple games. So some complete simple games are weighted and others are not.

In principle, complete simple games (or simple games) can be very complicated. Restricting to the subclass of symmetric (complete) simple games, where all agents have equal capabilities, simplifies things dramatically, as first found out in (May, 1952): All such games[1] are weighted, i.e., have a relatively simple structure.

In this paper we aim to generalize May's Theorem by providing a strategy to classify all classes of complete simple games, according to the agent's type composition (see Definition 5), with the property that every class member is weighted. It will turn out that this can happen only if at most five different types of agents are present and from all but one type there have to be very few agents, see lemmas 4 and 5.

Exact formulas for the number of sub classes of weighted games are rather rare. From May's Theorem, one can conclude that the number of weighted games with n agents all of the same type[2] is given by n. In (Kurz and Tautenhahn, 2013), the authors have presented an algorithm that can compute an exact enumeration formula for complete simple games with t types of agents and r, so-called, shift-minimal winning vectors depending on the number of agents n. No such algorithm is known for weighted games. Having our classification result at hand, we can enumerate the corresponding sub classes of weighted games since it will turn out, that in each case the number of occurring shift-minimal winning vectors is bounded by a small integer and the restrictions from the agent's type composition can be easily incorporated into the enumeration algorithm.

2. COMPLETE SIMPLE GAMES

A binary voting procedure can be modeled as a function $v : 2^N \to \{0,1\}$ mapping the coalition S of supporting agents to the aggregated group decision $v(S)$, where $N = \{1,\ldots,n\}$ and 2^N denotes the set of subsets of N. Quite naturally, several assumptions of a binary voting procedure are taken for granted:

(1) if no agent's supports the proposal, reject it;
(2) if all agent's supports the proposal, accept it;
(3) if the supporting clique of agents is enlarged by some additional agents, the group decision should not change from acceptance to rejection.

More formally, we state:

Definition 1. *A pair (v,N) is called simple game if N is a finite set, $v : 2^N \to \{0,1\}$ satisfies $v(\emptyset) = 0$, $v(N) = 1$, and $v(S) \le v(T)$ for all $S \subseteq T \subseteq N$.*

A more demanding assumption is to require that the agents are linearly ordered according to their capabilities to influence the final group decision. This can be formalized with the desirability relation introduced in (Isbell, 1956).

Definition 2. *Let (v,N) be a simple game. We write $i \sqsupset j$ (or $j \sqsubset i$) for two agents $i, j \in N$ if we have $v\Big(\{i\} \cup S\backslash\{j\}\Big) \ge v(S)$ for all $\{j\} \subseteq S \subseteq N\backslash\{i\}$ and we abbreviate $i \sqsupset j$, $j \sqsupset i$ by $i \square j$.*

The relation $\square$ partitions the set of agents N into equivalence classes $N_1,\ldots,N_t$.

Example 1. *For the weighted game $[4;5,4,2,2,0]$ we have $N_1 = \{1,2\}$, $N_2 = \{3,4\}$, and $N_3 = \{5\}$.*

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[1]More precisely, May's Theorem applies to (complete) simple games with one type of agents (see Section 2).

[2]They all have the form $[q;1,\ldots,1]$ with $q \in \{1,\ldots,n\}$.

Agents having the same weight are contained in the same equivalence class, while the converse is not necessarily true. But there always exists a different weighted representation of the same game such that the agents of each equivalence class have the same weight. For Example 1, such a representation is e.g. given by $[2;2,2,1,1,0]$.

Definition 3. *A simple game (v,N) is called complete if the binary relation $\sqsupset$ is a total preorder, i.e.,*

(1) *$i \sqsupset i$ for all $i \in N$,*
(2) *$i \sqsupset j$ or $j \sqsupset i$ for all $i,j \in N$, and*
(3) *$i \sqsupset j$, $j \sqsupset h$ implies $i \sqsupset h$ for all $i,j,h \in N$.*

All weighted games are obviously simple and complete.

Definition 4. *For a simple game (v,N) a coalition $S \subseteq N$ is called winning if $v(S) = 1$ and losing otherwise. If $v(S) = 1$, $v(T) = 0$ for all $T \subsetneq S$, then S is called minimal winning. A coalition with $v(S) = 0$, $v(T) = 1$ for all $S \subsetneq T \subseteq N$ is called maximal losing.*

In Example 1, coalition $\{2,3\}$ is winning, $\{2\}$ is minimal winning, $\{3\}$ is losing, and $\{3,5\}$ is maximal losing.

Definition 5. *For a complete simple game (v,N), the vector $(n_1,\ldots,n_t) \in \mathbb{N}^t_{>0}$, where $n_i = |N_i|$, is called type composition. The number t of equivalence classes is called number of types (of agents).*

The type composition of Example 1 is given by $\big(2,2,1\big)$ consisting of three types. Agents within the same equivalence class are interchangeable, i.e., since coalition $\{2,3\}$ is winning also the coalitions $\{1,3\}$, $\{1,4\}$, and $\{2,4\}$ have to be winning. The combinatorial explosion of the set of corresponding winning coalitions can be partially captured by:

Definition 6. *Let (v,N) be a complete simple game with type composition $(n_1,\ldots,n_t)$. Each vector $s = (s_1,\ldots,s_t) \in \mathbb{N}^t$ with $0 \le s_i \le n_i$ for all $1 \le i \le t$ is called coalition vector of (v,N). Coalition vector s is winning if we have $v(S) = 1$ for coalitions $S \subseteq N$ with $|S \cap N_i| = s_i$ for all $1 \le i \le t$, and losing otherwise.*

The just mentioned four winning coalitions can be condensed to the winning vector $(1,1,0)$.

Definition 7. *For two vectors $a = (a_1,\ldots,a_t) \in \mathbb{N}^t$ and $b = (b_1,\ldots,b_t) \in \mathbb{N}^t$ we write $a \le b$ if $a_i \le b_i$ for all $1 \le i \le t$.*

If a is a winning vector of a complete simple game and $a \le b$, then b is winning too. Next, we define a tightening of the concept of minimal winning and maximal losing coalitions for coalition vectors. To this end we have to assume $1 \sqsupset 2 \sqsupset \cdots \sqsupset n$ in the following.

Definition 8. *For two vectors $a = (a_1,\ldots,a_t) \in \mathbb{N}^t$ and $b = (b_1,\ldots,b_t) \in \mathbb{N}^t$ we write $a \preceq b$ if $\sum_{j=1}^{i} a_j \le \sum_{j=1}^{i} b_j$ for all $1 \le i \le t$. Vector a is called shift-minimal winning (SMW), if a is winning and all $b \preceq a$, with $b \ne a$, are losing. Similarly, vector a is called shift-maximal losing (SML), if a is losing and all $b \succeq a$, with $b \ne a$, are winning.*

An example is given by $(0,1,0) \preceq (1,0,0)$. The SMW vectors of Example 1 are given by $(1,0,0)$ and $(0,2,0)$. The unique SML vector is given by $(0,1,1)$. We remark that each complete simple game is uniquely characterized by its type composition and its full list of SMW vectors. Of course, not every collection of coalition vectors for a given type composition is a feasible set of SMW vectors.

Definition 9. *Let $a=(a_1,\dots,a_t)\in\mathbb{N}^t$ and $b=(b_1,\dots,b_t)\in\mathbb{N}^t$ be two vectors. We write $a\bowtie b$ if neither $a\preceq b$ nor $a\succeq b$, i.e., when they are incomparable. Mimicking the lexicographic order, we write $a > b$ if there exists an index $k\in\{0,\dots,n-1\}$ such that $a_j=b_j$ for all $1\le j\le k$ and $a_{j+1}>b_{j+1}$.*

A parameterization theorem for complete simple games with t types of agents has been given in (Carreras and Freixas, 1996):

Theorem 1.

(a) *Let vector $\widehat{n}=(n_1,\dots,n_t)\in\mathbb{N}^t_{>0}$ and a matrix*

$$\mathcal{S}=\begin{pmatrix} s_{1,1} & s_{1,2} & \dots & s_{1,t}\\ s_{2,1} & s_{2,2} & \dots & s_{2,t}\\ \vdots & \ddots & \ddots & \vdots\\ s_{r,1} & s_{r,2} & \dots & s_{r,t}\end{pmatrix}=\begin{pmatrix}\widehat{s}_1\\ \widehat{s}_2\\ \vdots\\ \widehat{s}_r\end{pmatrix}$$

satisfy the following properties:

(i) $0\le s_{i,j}\le n_j$, $s_{i,j}\in\mathbb{N}$ *for* $1\le i\le r$, $1\le j\le t$,
(ii) $\widehat{s}_i\bowtie\widehat{s}_j$ *for all* $1\le i<j\le r$,
(iii) *for each* $1\le j<t$ *there is at least one row-index i such that* $s_{i,j}>0$, $s_{i,j+1}<n_{j+1}$ *if* $t>1$ *and* $s_{1,1}>0$ *if* $t=1$, *and*
(iv) $\widehat{s}_i > \widehat{s}_{i+1}$ *for* $1\le i<r$.

Then, there exists a complete simple game (v,N) associated to $(\widehat{n},\mathcal{S})$.

(b) *Two complete simple games $(\widehat{n}_1,\mathcal{S}_1)$ and $(\widehat{n}_2,\mathcal{S}_2)$ are isomorphic if and only if $\widehat{n}_1=\widehat{n}_2$ and $\mathcal{S}_1=\mathcal{S}_2$.*

Besides being rather technical, there is some easy interpretation for the stated conditions. Condition (i) simply states that the $\widehat{s}_i$ are feasible with respect to the type composition $\widehat{n}$. If we would not have $\widehat{s}_i\bowtie\widehat{s}_j$, then either $\widehat{s}_i\preceq\widehat{s}_j$ or $\widehat{s}_i\succeq\widehat{s}_j$, so that one of both vectors can not be shift-minimal. Condition (iii) is necessary to enforce equivalence classes according to $\widehat{n}$ and condition (iv) prevents from row permutations. We call two complete simple games isomorphic if there exists a bijection for the respective agent's names preserving winning and losing coalitions.

Definition 10. *A simple game (v,N) is called weighted if and only if there exist weights $w_i\in\mathbb{R}_{\ge0}$, for all $i\in N$, and a quota $q\in\mathbb{R}_{>0}$ such that $v(S)=1$ is equivalent to $\sum_{i\in S}w_i\ge q$ for all $S\subseteq N$.*

3. (NON-) WEIGHTEDNESS

We have mentioned in the introduction that some complete simple games are weighted while others are not. In this section, we want to provide a method to decide which case occurs[3].

[3]Several algorithms to decide whether a given simple game is weighted or not are known in the literature, see e.g. (Taylor and Zwicker, 1999) for an overview.

Example 2. *Let* $\widehat{n} = (2,4)$ *and* $\mathcal{S} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$*, then the complete simple game* $(\widehat{n}, \mathcal{S})$ *is not weighted.*

Similar to the matrix $\mathcal{S}$ of the shift-minimal vectors, one can write down a matrix $\mathcal{L}$ of the shift-maximal losing vectors. If running time is not an issue, this can be easily done algorithmically:

```
For each coalition vector a=(a_1,...a_t)
   determine whether a is winning or losing
End
For each losing vector a=(a_1,...a_t)
  ok=True
  For i from 1 to t
      If a_i<n_i and a+e_i is losing
      Then ok=False
  End
  If ok==True Then output a
End
```

Here e_i denotes the ith unit vector and we have $\mathcal{L} = \begin{pmatrix} 1 & 2 \end{pmatrix}$ in Example 2.

Lemma 1. *For a complete simple game* (v,N) *let* $\tilde{S}$ *be a matrix of (some) winning vectors and* $\tilde{L}$ *be a matrix of (some) losing vectors. If there exist (row) vectors* x, y *with non-negative real entries,* $\|x\|_1 = \|y\|_1 > 0$ *and* $x\tilde{S} \le y\tilde{L}$*, then* (v,N) *can not be weighted.*

Proof. Combining the fact that the weight of each winning coalition is larger than the weight of each losing coalitions with the weighting of coalitions induced by x and y gives a contradiction. □

A well known fact from the literature is the inverse statement, i.e., for each non-weighted (complete) simple game there exists a set of winning coalitions (or winning vectors) and a set of losing coalitions (or losing vectors) with multipliers x, y certifying non-weightedness. The underlying concepts are trading transforms, see (Taylor and Zwicker, 1999), or dual multipliers in the theory of linear programming. An example of a non-weighted complete simple game can be extended to other type compositions:

Lemma 2. *Let* $G_1 = (v,N)$ *be a complete simple game with type composition* $\widehat{n} = (n_1, \ldots, n_t)$ *and* $\tilde{S}$ *a matrix of winning vectors,* $\tilde{L}$ *a matrix of losing vectors, and* x, y *be vectors according to Lemma 1, which certify non-weightedness of* (v,N)*. For each vector* $\widehat{m} \ge \widehat{n}$ *there exists a non-weighted complete simple game* $G_2 = (v', N')$ *with type composition* $\widehat{m}$*.*

Proof. We choose N' such that $N \subseteq N'$ and set $v'(S) = v(S)$ for all $S \subseteq N$, i.e., winning vectors of G_1 are also winning in G_2 and losing vectors of G_1 are also losing in G_2. All coalition vectors of G_2 that are comparable to the already assigned vectors, i.e., to the coalition vectors of G_1, are accordingly set to be either winning or losing. For the remaining vectors we have some freedom, but for simplicity determine them to be losing vectors. We can easily check that G_2 is completely characterized and is indeed a complete simple game. Since the rows of $\tilde{S}$ are also winning vectors in G_2 and the rows of $\mathcal{L}$ are also losing vectors in G_2, we can apply Lemma 1 with the original vectors x, y to deduce that G_2 is non-weighted. □

As an example, let (v,N) be uniquely characterized by $\widehat{n}=(3,4)$ and $\mathcal{S}=\begin{pmatrix}2 & 2\end{pmatrix}$. The matrix of shift-maximal losing vectors is given by $\mathcal{L}=\begin{pmatrix}3 & 0\\ 1 & 4\end{pmatrix}$ and we have $x=(2)$, $y=(1,1)$ as a certificate for non-weightedness. For $\widehat{m}=(6,6)$ the construction of Lemma 2 gives the game with type composition $\widehat{m}$ and $\mathcal{S}=\begin{pmatrix}2 & 2\end{pmatrix}$. Now the matrix of shift-maximal losing vectors is given by $\mathcal{L}=\begin{pmatrix}3 & 1 & 0\\ 0 & 4 & 6\end{pmatrix}^T$. From the reused vectors x, y we can conclude the non-weightedness of the larger complete simple game. The degree of freedom in the proof of Lemma 2 allows us to also conclude that the complete game given by type composition $\widehat{m}$ and $\mathcal{S}=\begin{pmatrix}2 & 2\\ 0 & 6\end{pmatrix}$ is also non-weighted. Here we have $\mathcal{L}=\begin{pmatrix}3 & 0\\ 1 & 4\end{pmatrix}$. The common parts $\tilde{S}$ and $\tilde{L}$ have to be chosen accordingly.

Definition 11. *We call a type composition $\widehat{n}=(n_1,\dots,n_t)$ weighted if all complete simple games, given by a matrix $\mathcal{S}$ of its SMW vectors and $\widehat{n}$, are weighted. Otherwise we call $\widehat{n}$ non-weighted.*

Since $(3,4)\geq(2,4)$ we do not learn anything new, i.e., Lemma 2 alone is sufficient to prove:

Lemma 3. *Each type composition $\widehat{n}=(n_1,n_2)$ with $n_1\geq 2$ and $n_2\geq 4$ is non-weighted*

Lemma 4. *Each type composition $\widehat{n}=(n_1,\dots,n_t)\in\mathbb{N}_{>0}^t$ with $t\geq 6$ is non-weighted.*

Proof. The game $(\widehat{m},\mathcal{S})$ with

$$\mathcal{S}=\begin{pmatrix}1&1&0&0&0&0\\1&0&1&0&0&1\\1&0&0&1&1&0\\0&1&1&1&0&0\\0&0&1&1&1&1\end{pmatrix}$$

is complete, non-weighted, and has $\widehat{m}=(1,1,1,1,1,1)$ as its type composition with 6 types. For $t>6$ we consider the complete simple non-weighted game with type composition with $\widehat{m}=(1,\dots,1)\in\mathbb{N}_{>0}^t$ uniquely characterized by its matrix

$$\mathcal{S}=\begin{pmatrix}1&1&0&0&0&0&|&1&0&1&0&\dots\\1&0&1&0&0&1&|&0&1&0&1&\dots\\1&0&0&1&1&0&|&1&0&1&0&\dots\\0&1&1&1&0&0&|&0&1&0&1&\dots\\0&0&1&1&1&1&|&1&0&1&0&\dots\end{pmatrix}$$

of SMW vectors. Lemma 2 transfers the result to arbitrary type compositions $\widehat{n}$ with $t\geq 6$ types. □

With the help of lemmas 2 and 4, we can propose the following strategy to classify all weighted type compositions $\widehat{n}$. For small n, determine all complete simple games with at most 5 types of agents and determine which ones are weighted or non-weighted. Taking only the smallest examples, we obtain a generalized version of Lemma 3 and may hope that all other cases correspond to weighted type compositions. Doing that we obtain:

Lemma 5. *For each type composition* $\widehat{n} = (2,4)$, $(2,2,2)$, $(1,1,5)$, $(1,2,3)$, $(1,3,2)$, $(2,1,4)$, $(2,4,1)$, $(1,1,1,3)$, $(1,1,3,1)$, $(2,1,2,1)$, $(2,3,1,1)$, $(1,2,2,1)$, $(1,2,1,2)$, $(1,1,2,2)$, $(1,2,1,1,1)$, $(1,1,2,1,1)$, $(1,1,1,2,1)$, $(1,1,1,1,2)$ *there exists a non-weighted complete simple game attaining* $\widehat{n}$.

Conjecture 1. *Each type composition* $\widehat{n}$ *is either weighted or there exists a type composition* $\widehat{m}$*, contained in the list of Lemma 5, with* $\widehat{n} \geq \widehat{m}$.

In the next section, we propose an algorithmic approach to prove Conjecture 1. As an example, we prove some special cases in Subsection 4.5.

4. WEIGHTED TYPE COMPOSITIONS

Due to Lemma 3, for $t = 2$ types of agents, the only possible candidates for weighted type compositions are of the form $(1,\star)$, $(\star,1)$, $(\star,2)$, and $(\star,3)$, where $\star$ stands for an arbitrary positive integer.

Definition 12. *A set* $\omega = (n_1,\dots,n_{i-1},\star,n_{i+1},\dots,n_t)$ *of type compositions is called* $i\star$ *family with* t *types. We call* ω *weighted if all of its elements are weighted.*

Corollary 1. *(of Conjecture 1)* $(\star)$, $(1,\star)$, $(\star,1)$, $(\star,2)$, $(\star,3)$, $(\star,1,1)$, $(\star,1,2)$, $(\star,1,3)$, $(\star,2,1)$, $(\star,3,1)$, $(1,\star,1)$, $(1,1,4)$, $(1,2,2)$, $(\star,1,1,1)$, $(\star,1,1,2)$, $(\star,2,1,1)$, $(1,\star,1,1)$, $(1,1,2,1)$, *and* $(\star,1,1,1,1)$ *are weighted.*

In this section, we propose an algorithm capable to prove that the type compositions of a given $i\star$ family ω with t types are weighted (if true).

4.1. **Step 1: SMW Vectors.** Consider a complete simple game with type composition $\widehat{n} \in \omega$ and matrix $\mathcal{S}$ of its SMW vectors. We aim to write down a finite set of parameterized candidates for the rows of $\mathcal{S}$ only depending on ω. With $\alpha = \big\{a = (a_1,\dots,a_{i-1}) \mid 0 \leq a_j \leq n_j,\, 1 \leq j \leq i-1\big\}$ we introduce the injective function $\tau : \alpha \to \mathbb{N}$ by $\tau(a) = \sum_{j=1}^{i-1} a_i \prod_{k=j+1}^{i-1}(n_k+1)$, i.e., we are just numbering the elements of α in a convenient way. Additionally we set $\tau(\omega) := \tau\big((n_1,\dots,n_{i-1})\big) + 1 = |\alpha|$.

Lemma 6. *Given an* $i\star$ *family* ω *with* t *types let* $C(\omega)$

$$= \big\{ \big(a, m_{\tau(a)} - c, b\big) \mid a \in \alpha, b \in \beta, c \in \mathbb{N}, c \leq \Lambda \big\},$$

where $\beta = \big\{(b_{i+1},\dots,b_n) \in \mathbb{N}^{n-i} \mid b_h \leq n_h\big\}$, $\Lambda = \sum_{h=i+1}^{n} n_h$, *and the* m_j *are free variables. For each simple game with type composition* $\widehat{n} \in \omega$ *and matrix* $\mathcal{S} = (\tilde{s}_1,\dots,\tilde{s}_r)^T$ *of its SMW vectors, there exists an allocation of the* m_j*, such that* $\tilde{s}_h \in C(\omega)$ *for all* $1 \leq h \leq r$.

Proof. Given a complete simple game with type composition $\widehat{n} \in \omega$ and matrix $\mathcal{S} = (\tilde{s}_1,\dots,\tilde{s}_r)^T$ of its SMW vectors. For each $a \in \alpha$ let $\tilde{t}_a = (a,m,b)$, where $b = (b_{i+1},\dots,b_n) \in \beta$, the row-vector of $\mathcal{S}$ whose first $i-1$ coordinates coincide with a and whose ith coordinate $m \in \mathbb{N}$ is maximal. If $\tilde{t}_a$ exists, we set $m_{\tau(a)} = m$ and $m_{\tau(a)} = -1$ otherwise. Now let $\tilde{s}_j = (a,m',b')$ be an arbitrary SMW vector whose first $i-1$ components coincide with a. Clearly $m' \in \mathbb{N}$, $m' \leq m$, and $b' = (b'_{i+1},\dots,b'_n) \in \beta$. If $m' < m$ then there exists an index $k \in \{i+1,\dots,n\}$ with $m' + \sum_{h=i+1}^{k} b'_h > m + \sum_{h=i+1}^{k} b_h$ since $\sum_{h=1}^{i-1} a_h + m' < \sum_{h=1}^{i-1} a_h + m$ and $\tilde{t}_a \bowtie \tilde{s}_j$. With $\sum_{h=i+1}^{k} b_h \geq 0$ and $\sum_{h=i+1}^{k} b'_h \leq \sum_{h=i+1}^{n} n_h$ we have $m' > n - \Lambda$. $\square$

Thus we can parameterize the potential SMW vectors of a complete simple game attaining ω using at most $\tau(\omega)$ parameters as elements in $C(\omega)$, where

$$|C(\omega)| \le \left(\sum_{j=1, j\neq i}^{n} n_j\right) \cdot \prod_{j=1, j\neq i}^{n} (n_j+1), \tag{1}$$

i.e., the number rows r is bounded by ω.

4.2. **Step 2: Matrices of All Shift-minimal Winning Vectors.** Given an $i\star$ family ω with t types, the sets of SMW vectors of a complete simple game with type composition $\widehat{n} \in \omega$ are subsets of $C(\omega)$. Thus, we can loop over all elements of $2^{C(\omega)}$ and need to check whether the selected subsets satisfy the conditions of Theorem 1(a). The technical difficulty we have to face here is, that the entries can linearly depend on the parameters m_j. In (Kurz and Tautenhahn, 2013) the similar situation, where all entries $s_{i,j}$ are parameters, has been treated. There it is shown that all feasible cases, meeting the conditions of Theorem 1(a), can be formulated as a union of systems of linear inequality systems in terms of the parameters. Exemplarily, condition (a)(ii) is satisfied for a pair of indices $1 \le i < j \le r$, if two further indices $1 \le h, k \le t$ exist with

$$\sum_{u=1}^{h} s_{i,u} + 1 \le \sum_{u=1}^{h} s_{j,u} \text{ and } \sum_{u=1}^{k} s_{i,u} \ge 1 + \sum_{u=1}^{k} s_{j,u}.$$

Performing these steps yields a finite list $\mathcal{S}_1, \mathcal{S}_2, \dots$ of matrices, whose entries are linear functions of the parameters m_j, such that the rows of each matrix $\mathcal{S}_h$ are the (parametric) SMW vectors of complete simple games with type composition in ω whenever the parameters m_i satisfy the linear constraints of the corresponding polytope P_h. Moreover, all complete simple games with type composition in ω are captured by one of the pairs $\mathcal{S}_h$, P_h. (It is indeed possible to obtain a partition of the desired space.)

4.3. **Step 3: Losing Vectors.** For simplicity, we assume that we are given a single pair $(\mathcal{S}_h, P_h)$ according to Subsection 4.2. In (Kurz and Tautenhahn, 2013) the parametric Barvinok algorithm was applied to count the respective number of complete simple games. Here we want to study weightedness so that we also need a description (not necessarily the most compact description) of the set of losing vectors. To this end, we mention that the SML vectors of a complete simple game are either incomparable to all SMW vectors, and so contained in $C(\omega)$, or arise as so-called shifts of one of the SMW vectors, i.e., special vectors that have a fairly small $\|\cdot\|_1$-distance to one of the SMW vectors. Due to space limitations we just mention, that it is possible to exactly describe a set of all candidates for SML vectors, similar as $C(\omega)$ for the set of SMW vectors. Then we can again consider subsets of the set of candidates and have to check that the implications, with respect to be a winning or a losing vector, are non-contradicting and that the state of each vector can be deduced in any case. If properly implemented with all technical details, things boil down to a splitting of a sub case $(\mathcal{S}_h, P_h)$ into a finite list of sub sub cases

$$(\mathcal{S}_h, \mathcal{L}_{h,1}, P_{h,1}), (\mathcal{S}_h, \mathcal{L}_{h,2}, P_{h,2}), \dots,$$

where the rows of the $\mathcal{L}_{h,j}$ correspond to (not necessarily shift-maximal) losing vectors and the $P_{h,j}$ are sub polytopes of P_h.

4.4. **Step 4: Weighted Representation.** For simplicity, we assume that we are given a single triple $\Gamma = \left(\mathcal{S}_h, \mathcal{L}_h, P_h\right)$ according to Subsection 4.3. For each integral choice of the parameters m_j in P_h, we have a unique complete simple game at hand and can check whether it is weighted with the help of a linear program, see e.g. (Taylor and Zwicker, 1999). If at least one of such games is non-weighted, then we can use the methods of Section 3 to deduce that a certain class of type compositions is non-weighted. So let us assume that all games corresponding to Γ are indeed weighted.

Thus, for each (of the possibly infinitely many) complete simple games corresponding to Γ, there exist feasible weights obtained at a basis solution of the corresponding linear program, which is uniquely determined by Γ but depends on the parameters m_i. Nevertheless, the number of possible basis solutions is finite. Thus, we can loop over all possible parametric basis solutions δ_j and determine the corresponding list of polytopes $P_{h,k}$, such that δ_j yields a feasible weighting of the complete simple games corresponding to $\mathcal{S}_h, \mathcal{L}_h$ whenever the parameters m_l are in $P_{h,k}$. If $\cup_k P_{h,k} = P_h$, then ω is weighted for sub case Γ.

4.5. **Examples.** In the previous four subsections we have sketched an algorithm that is capable to prove that a given $i\star$ family ω with t types is weighted (if the statement is indeed true). Due to space limitations, we have not given all technical, sometimes non-trivially, details. Instead, we want to give examples for special cases.

Lemma 7. $\omega = (\star)$ *is weighted.*

Proof. According to Step 1 we have $\tau(\omega) = 1$ parameter m_0, $\Lambda = 0$, and $C(\omega) = \{(m_0)\}$ with $|C(\omega)| = 1$. Since each complete simple game consists of a least one shift-minimal winning coalition we obtain the one-element list $\Big(\mathcal{S}_1 = (m_0), P_1 = \{(m_0) \in \mathbb{R}^1 \mid 1 \le m_0 \le n_1\}\Big)$. Since all elements of $C(\omega)$ have already been assigned to be shift-minimal winning, there remains the unique shifted vector $(m_0 - 1)$ to be losing in Step 3. In Step 4, we can obtain the basis solution $q = m_0$, $w_1 = 1$, which is feasible for the entire polytope P_1. (We only state the weights for each type of agents, numbered from 1 to t.) □

Lemma 8. $\omega = (1, \star)$ *is weighted.*

Proof. According to Step 1 we have $\tau(\omega) = 2$ parameters m_0, m_1, $\Lambda = 0$, and $C(\omega) = \{(0, m_0), (1, m_1)\}$ with $|C(\omega)| = 2$. In Step 2 we have to consider the three non-empty subsets of $C(\omega)$. For the case $\mathcal{S} = \begin{pmatrix} 0 & m_0 \end{pmatrix}$ we observe that condition (a)(iii) of Theorem 1 can not be met for any allocation of the parameters m_0, m_1. Thus, there remain only two cases with non-empty polytopes for the parameters:

$$\mathcal{S}_1 = \begin{pmatrix} 1 & m_1 \end{pmatrix}, P_1 = \{(m_1) \in \mathbb{R} \mid 0 \le m_1 \le n_2 - 1\}$$

and

$$\mathcal{S}_2 = \begin{pmatrix} 1 & m_1 \\ 0 & m_0 \end{pmatrix}, P_2 = \left\{ \begin{pmatrix} m_0 \\ m_1 \end{pmatrix} \in \mathbb{R}^2 \mid \begin{matrix} m_1 \ge 0 \\ m_1 + 2 \le m_0 \\ m_0 \le n_2 \end{matrix} \right\}.$$

In Step 3, each of the two sub cases is split into two sub sub cases $\Gamma_1 = (\mathcal{S}_1, \mathcal{L}_{1,1}, P_{1,1})$, $\Gamma_2 = (\mathcal{S}_1, \mathcal{L}_{1,2}, P_{1,2})$, $\Gamma_3 = (\mathcal{S}_2, \mathcal{L}_{2,1}, P_{2,1})$, $\Gamma_3 = (\mathcal{S}_2, \mathcal{L}_{2,2}, P_{2,2})$, with $\mathcal{L}_{1,1} = \begin{pmatrix} 1 & m_1 - 1 \\ 0 & n_2 \end{pmatrix}$,

$\mathcal{L}_{1,2} = \begin{pmatrix} 0 & n_2 \end{pmatrix}$, $\mathcal{L}_{2,1} = \begin{pmatrix} 1 & m_1 - 1 \\ 0 & m_0 - 1 \end{pmatrix}$, $\mathcal{L}_{2,2} = \begin{pmatrix} 0 & m_0 - 1 \end{pmatrix}$,

$$
\begin{aligned}
P_{1,1} &= \{(m_1) \in \mathbb{R} \mid m_1 \geq 1\} \cap P_1, \\
P_{1,2} &= \{(m_1) \in \mathbb{R} \mid m_1 = 0\} \cap P_1, \\
P_{2,1} &= \left\{(m_0, m_1) \in \mathbb{R}^2 \mid m_1 \geq 1\right\} \cap P_2, \\
P_{2,2} &= \left\{(m_0, m_1) \in \mathbb{R}^2 \mid m_1 = 0\right\} \cap P_2.
\end{aligned}
$$

In Step 4, fortunately, no further splitting is necessary, and we can even condense sub cases. For Γ_1, Γ_2 we have the weighted representation $[n_2 + 1; n_2 + 1 - m_1, 1]$ and for Γ_3, Γ_4 we have the weighted representation $[m_0; m_0 - m_1, 1]$. □

Using the parametric Barvinok algorithm or elementary summation formulas with case differentiation, we conclude the well known fact that the number of n-agent weighted games with type composition $(\star)$ is n. Similarly, we conclude, that the number of n-agent weighted games with type composition $(1, \star)$ is given by

$$
\frac{n^3 - n}{6} = \frac{n(n-1)(n+1)}{6} = \binom{n+1}{3}
$$

for all $n \in \mathbb{N}$.

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APPENDIX

Non-weighted Examples. In order to prove Lemma 5 it suffices to give a non-weighted example for each case. Those can easily be found by classifying all complete simple games with up to 7 agents. That $(2,4)$ is non-weighted has already been observed in Section 3. For the other type compositions mentioned in Lemma 5, we provide the following examples:

- $\widehat{n} = (2,2,2)$, $\mathcal{S} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 2 \end{pmatrix}$, $\tilde{L} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, $x = (1,0,1)$, $y = (2)$.

- $\widehat{n}=(1,1,5)$, $\mathcal{S}=\begin{pmatrix}1&0&2\\0&1&3\end{pmatrix}$, $\tilde{L}=\begin{pmatrix}1&1&0\\0&0&5\end{pmatrix}$, $x=(1,1)$, $y=(1,1)$.
- $\widehat{n}=(1,2,3)$, $\mathcal{S}=\begin{pmatrix}1&1&0\\0&1&3\end{pmatrix}$, $\tilde{L}=\begin{pmatrix}1&0&2\\0&2&1\end{pmatrix}$, $x=(1,1)$, $y=(1,1)$.
- $\widehat{n}=(1,3,2)$, $\mathcal{S}=\begin{pmatrix}1&0&2\\0&3&0\end{pmatrix}$, $\tilde{L}=\begin{pmatrix}1&1&0\\0&2&2\end{pmatrix}$, $x=(1,1)$, $y=(1,1)$.
- $\widehat{n}=(2,1,4)$, $\mathcal{S}=\begin{pmatrix}1&0&2\\0&1&3\end{pmatrix}$, $\tilde{L}=\begin{pmatrix}2&0&0\\0&0&4\end{pmatrix}$, $x=(2,0)$, $y=(1,1)$.
- $\widehat{n}=(2,4,1)$, $\mathcal{S}=\begin{pmatrix}2&0&1\\0&4&0\end{pmatrix}$, $\tilde{L}=\begin{pmatrix}1&2&1\end{pmatrix}$, $x=(1,1)$, $y=(2)$.
- $\widehat{n}=(1,1,1,3)$, $\mathcal{S}=\begin{pmatrix}1&0&0&2\\0&1&1&1\\0&1&0&3\end{pmatrix}$, $\tilde{L}=\begin{pmatrix}1&1&0&0\\0&0&1&3\end{pmatrix}$, $x=(1,1,0)$, $y=(1,1)$.
- $\widehat{n}=(1,1,3,1)$, $\mathcal{S}=\begin{pmatrix}1&0&1&1\\0&1&2&0\end{pmatrix}$, $\tilde{L}=\begin{pmatrix}1&1&0&0\\0&0&3&1\end{pmatrix}$, $x=(1,1)$, $y=(1,1)$.
- $\widehat{n}=(2,1,2,1)$, $\mathcal{S}=\begin{pmatrix}1&1&0&1\\1&0&2&0\end{pmatrix}$, $\tilde{L}=\begin{pmatrix}2&0&0&0\\0&1&2&1\end{pmatrix}$, $x=(1,1)$, $y=(1,1)$.
- $\widehat{n}=(2,3,1,1)$, $\mathcal{S}=\begin{pmatrix}1&1&0&1\\0&3&1&0\end{pmatrix}$, $\tilde{L}=\begin{pmatrix}2&0&0&0\\1&0&1&1\\0&3&0&1\end{pmatrix}$. The vectors $x=(3)$, $y=(1,1,1)$ yield $(3,3,0,3)\preceq(3,3,1,2)$, which is generally also sufficient to conclude non-weightedness, since the coalition vectors can be shifted accordingly. Here we may consider two times the SMW vector $(1,1,0,1)$ and one times the shifted winning vector $(1,0,1,1)$ to obtain $(3,3,1,2)$.
- $\widehat{n}=(1,2,2,1)$, $\mathcal{S}=\begin{pmatrix}1&0&1&1\\0&2&1&0\end{pmatrix}$, $\tilde{L}=\begin{pmatrix}1&1&0&0\\0&1&2&1\end{pmatrix}$, $x=(1,1)$, $y=(1,1)$.
- $\widehat{n}=(1,2,1,2)$, $\mathcal{S}=\begin{pmatrix}1&0&0&2\\0&2&1&0\\0&2&0&2\end{pmatrix}$, $\tilde{L}=\begin{pmatrix}1&1&0&0\\0&1&1&2\end{pmatrix}$, $x=(1,1,0)$, $y=(1,1)$.
- $\widehat{n}=(1,1,2,2)$, $\mathcal{S}=\begin{pmatrix}1&0&0&2\\0&1&1&1\end{pmatrix}$, $\tilde{L}=\begin{pmatrix}1&1&0&0\\0&1&0&2\\0&0&2&2\end{pmatrix}$, $x=(1,2)$, $y=(1,1,1)$.
- $\widehat{n}=(1,2,1,1,1)$, $\mathcal{S}=\begin{pmatrix}1&1&0&0&0\\1&0&1&1&0\\0&2&1&0&0\\0&1&1&1&1\end{pmatrix}$, $\tilde{L}=\begin{pmatrix}1&0&1&0&1\\0&2&0&1&0)\end{pmatrix}$, $x=(1,0,0,1)$, $y=(1,1)$.

- $\widehat{n} = (1,1,2,1,1)$, $\mathcal{S} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 \end{pmatrix}$, $\tilde{L} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$, $x = (1,0,0,1)$, $y = (1,1)$.
- $\widehat{n} = (1,1,1,2,1)$, $\mathcal{S} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}$, $\tilde{L} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$, $x = (1,0,0,1)$, $y = (1,1)$.
- $\widehat{n} = (1,1,1,1,2)$, $\mathcal{S} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$, $\tilde{L} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$, $x = (1,0,0,1)$, $y = (1,1)$.

Proofs of some Special Cases of Conjecture 1. For $n \le 5$ agents, all complete simple games are weighted, so that all type compositions $\widehat{n}$ with $\|\widehat{n}\|_1 \le 5$ are weighted, i.e., $(1,2,2)$ and $(1,1,2,1)$ are weighted. For $n = 6$ agents, we remark that exactly 1111 of the 1171 complete simple games are weighted, i.e., 60 games are non-weighted. For $n \ge 7$ agents, the discrepancy quickly increases. For $n = 7$ agents there are 44313 complete simple games, while only 29373 of them are weighted. For $n = 8$ agents, the respective counts are given by 16175188 and 2730164. Via exhaustive enumeration one can easily show that $(1,1,4)$ is weighted. The cases $(\star)$ and $(\star,1)$ have already been proven in Subsection 4.5.

Lemma 9. $\omega = (\star,1)$ *is weighted.*

Proof. According to Step 1 we have $\tau(\omega) = 1$ parameter m_0, $\Lambda = 1$, and $C(\omega) = \{(m_0,0),(m_0,1),(m_0-1,0),(m_0-1,1)\}$ with $|C(\omega)| = 4$. For Step 2 we remark that no pair of elements of $C(\omega)$ is incomparable, so that we have $r = 1$. There is no need to consider the cases $\mathcal{S} = (m_0-1,0)$ or $\mathcal{S} = (m_0-1,1)$ separately, since they are captured by the two cases $\mathcal{S} = (m_0,0)$ and $\mathcal{S} = (m_0,1)$. Generally, we can require that for each used parameter m_j there is a SMW vector taking the value $m_j - 0$ at the respective coordinate. Due to condition (a)(iii) of Theorem 1, $\mathcal{S} = (m_0,1)$ is not possible. Thus, there remains only one case with non-empty polytope for the parameter:

$$\mathcal{S}_1 = \begin{pmatrix} m_0 & 0 \end{pmatrix}, P_1 = \{(m_0) \in \mathbb{R} \mid 1 \le m_0 \le n_1\}.$$

In Step 3 there remains a single case: $\Gamma_1 = (\mathcal{S}_1,\mathcal{L}_1,P_1)$ with $\mathcal{L} = \begin{pmatrix} m_0-1 & 0 \end{pmatrix}$. In Step 4 we determine the weighted representation $[m_0;1,0]$ for Γ_1. □

Lemma 10. $\omega = (\star,2)$ *is weighted.*

Proof. According to Step 1 we have $\tau(\omega) = 1$ parameter m_0, $\Lambda = 2$, and $C(\omega) = \{(m_0,0),(m_0,1),(m_0,2),(m_0-1,0),(m_0-1,1),(m_0-1,2),(m_0-2,0),(m_0-2,1),(m_0-2,2)\}$ with $|C(\omega)| = 9$. As in the proof of Lemma 9, we remark that we can assume that one of the vectors $(m_0,0)$, $(m_0,1)$, or $(m_0,2)$ has to be chosen as a SMW vector. Only the first one is incomparable with one of the other vectors of $C(\omega)$. $\mathcal{S} = \begin{pmatrix} m_0 & 2 \end{pmatrix}$ is impossible due to condition (a)(iii) of Theorem 1. Thus Step 2 yields the cases

- $\mathcal{S}_1 = \begin{pmatrix} m_0 & 0 \end{pmatrix}$, $P_1 = \{(m_0) \in \mathbb{R} \mid 1 \leq m_0 \leq n_1\}$
- $\mathcal{S}_2 = \begin{pmatrix} m_0 & 1 \end{pmatrix}$, $P_2 = \{(m_0) \in \mathbb{R} \mid 1 \leq m_0 \leq n_1\}$
- $\mathcal{S}_3 = \begin{pmatrix} m_0 & 0 \\ m_0 - 1 & 2 \end{pmatrix}$, $P_3 = P_2 = P_1$

No splitting or modification of the P_i is necessary in Step 3, so that $\Gamma_i = (\mathcal{S}_i, \mathcal{L}_i, P_i)$, where $\mathcal{L}_1 = \begin{pmatrix} m_0 - 1 & 2 \end{pmatrix}$, $\mathcal{L}_2 = \begin{pmatrix} m_0 & 0 \\ m_0 - 1 & 2 \end{pmatrix}$, and $\mathcal{L}_3 = \begin{pmatrix} m_0 - 1 & 1 \end{pmatrix}$ for all $1 \leq i \leq 3$. In Step 4 we determine the weighted representations $[m_0; 1, 0]$ for Γ_1, $[2m_0 + 1; 2, 1]$ for Γ_2, and $[2m_0; 2, 1]$ for Γ_3. □

Lemma 11. $\omega = (\star, 3)$ *is weighted.*

Proof. According to Step 1, we have $\tau(\omega) = 1$ parameter m_0, $\Lambda = 3$, so that $|C(\omega)| = 16$. As in the two previous proofs, we remark that we can assume that one of the vectors $(m_0, 0)$, $(m_0, 1)$, $(m_0, 2)$, or $(m_0, 3)$ has to be chosen as a SMW vector. Here we can check that at most $r = 2$ elements of $C(\omega)$ can be pairwise incomparable. Excluding $\mathcal{S} = \begin{pmatrix} m_0 & 3 \end{pmatrix}$ with condition (a)(iii) of Theorem 1, there remain the following possibilities for $r = 1$:

- $\mathcal{S}_1 = \begin{pmatrix} m_0 & 0 \end{pmatrix}$, $P_1 = \{(m_0) \in \mathbb{R} \mid 1 \leq m_0 \leq n_1\}$
- $\mathcal{S}_2 = \begin{pmatrix} m_0 & 1 \end{pmatrix}$, $P_2 = P_1$
- $\mathcal{S}_3 = \begin{pmatrix} m_0 & 2 \end{pmatrix}$, $P_3 = P_1$

For $r = 2$ we additionally obtain:

- $\mathcal{S}_4 = \begin{pmatrix} m_0 & 0 \\ m_0 - 1 & 2 \end{pmatrix}$, $P_4 = P_1$
- $\mathcal{S}_5 = \begin{pmatrix} m_0 & 0 \\ m_0 - 1 & 3 \end{pmatrix}$, $P_5 = P_1$
- $\mathcal{S}_6 = \begin{pmatrix} m_0 & 1 \\ m_0 - 1 & 3 \end{pmatrix}$, $P_6 = P_1$
- $\mathcal{S}_7 = \begin{pmatrix} m_0 & 0 \\ m_0 - 2 & 3 \end{pmatrix}$, $P_7 = [2, n_1]$

For $i \in \{1, 2, 5, 6, 7\}$ no splitting of the P_i is necessary in Step 3, so that $\Gamma_i = (\mathcal{S}_i, \mathcal{L}_i, P_i)$, where $\mathcal{L}_1 = \begin{pmatrix} m_0 - 1 & 3 \end{pmatrix}$, $\mathcal{L}_2 = \begin{pmatrix} m_0 & 0 \\ m_0 - 1 & 3 \end{pmatrix}$, $\mathcal{L}_5 = \begin{pmatrix} m_0 - 1 & 2 \end{pmatrix}$, $\mathcal{L}_6 = \begin{pmatrix} m_0 & 0 \\ m_0 - 1 & 2 \end{pmatrix}$, $\mathcal{L}_7 = \begin{pmatrix} m_0 - 1 & 1 \end{pmatrix}$. For the remaining indices $i \in \{3, 4\}$ cases split up as $\Gamma_{i,j} = (\mathcal{S}_i, \mathcal{L}_{i,j}, P_{i,j})$, where

- $\mathcal{L}_{3,1} = \begin{pmatrix} m_0 & 1 \\ m_0 - 1 & 3 \end{pmatrix}$, $P_{3,1} = \{n_1\}$
- $\mathcal{L}_{3,2} = \begin{pmatrix} m_0 + 1 & 0 \\ m_0 - 1 & 3 \end{pmatrix}$, $P_{3,2} = [1, n_1 - 1]$
- $\mathcal{L}_{4,1} = \begin{pmatrix} m_0 - 1 & 1 \end{pmatrix}$, $P_{3,1} = \{1\}$
- $\mathcal{L}_{4,2} = \begin{pmatrix} m_0 - 1 & 1 \\ m_0 - 2 & 3 \end{pmatrix}$, $P_{3,2} = [2, n_1]$

In Step 4 we obtain the weighted representations

- $[m_0; 1, 0]$ for Γ_1
- $[3m_0 + 1; 3, 1]$ for Γ_2
- $[3m_0 + 4; 3, 2]$ for $\Gamma_{3,1}$ and $\Gamma_{3,2}$

- $[2m_0; 2, 1]$ for $\Gamma_{4,1}$ and $\Gamma_{4,2}$
- $[3m_0; 3, 1]$ for Γ_5
- $[2m_0 + 1; 2, 1]$ for Γ_6
- $[3m_0; 3, 2]$ for Γ_7

□

Thus, Conjecture 1 and Corollary 1 are proven for all type compositions with t types, where either $t < 3$ or $t > 5$. For $t \in \{3, 4, 5\}$, the necessary case differentiations become more and more complex so that we aim for a computer aided proof.

Open Problems. Of course, Conjecture 1 has to be proven in the first run. Several ways to extend these considerations are imaginable. One can ask for a similar classification for roughly-weighted games, i.e., for games where the weight of a winning coalition can equal the weight of another losing coalition, or for complete simple games of small dimension, i.e., games which can be represented as the intersection of a small number of weighted games. A possibly more challenging open problem is to provide explicit enumeration results for other sub classes of weighted games, than the ones known in the literature or those that can be concluded from the line of consideration presented in this paper.

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