University of Bayreuth
Faculty for Mathematics and Physics

Polyominoes with maximum convex hull

Sascha Kurz

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advised by Prof. Dr. A. Kerber
University of Bayreuth
and by Prof. Dr. H. Harborth
Technical University of Braunschweig

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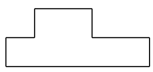
Preface

The first time I came along with **polyominoes** was in 1998 when I read a do-it-yourself story about a little worm named Heiner Wormeling [118]. I am going to tell a short version of this story in my own words:

Heiner Wormeling, his wife Amelia and baby Wermentrude just recovered their procession into a new lair. But Amelia was not amused seeing the bath room. “Heiner! Come to me!” Heiner reluctantly wormed one’s way towards the bath room leaving his comfortable armchair.“My dear, what’s wrong?” “Didn’t the builder promised to tile the whole bath? NOTHING, NOTHING is done yet and in the corner is standing a big box with tiles!” “I’ll phone him.” The builder apologized “Sorry chief, we had a little problem. Have a look at the funny tiles your wife ordered, we can’t match them without leaving spaces.” Heiner mashed “That’s ridiculous! Why didn’t you form a rectangle?” “That’s exactly what we tried, but without success.”. “Ridiculous! I’ll do it by myself!”.

Are you able to do it? Here is the tile  .

Two days later Heiner gave up and called his friend Albert Wormstone who works for the patent office. After hearing Heiner’s story Albert said “Your tile is some kind of a polyomino, that's a plane figure of equally sized squares neighboring edge-to-edge. In 1969 Klarner defined the order of a polyomino as the minimum number of copies of a polyomino filling a rectangle." Albert told Heiner that his polyomino has order 78 and gave him a solution to fill a rectangle. Heiner ran into the kitchen and said proudly "Amelia, I have solved the tile problem. All I have to do is taking 78 tiles and build a rectangle." "Wish you a lot of fun", Amelia replied. Heiner went into the bathroom and had a look at the description of the box.

Combinatoric Ceramic Factory
Heptomino Tiles
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During the winter semester 2002/2003 I took part in a course called "Discrete Geometry" lectured by Prof. Dr. H. Harborth at the Technical University of Braunschweig. A lot of unsolved problems concerning the field of Discrete Geometry were discussed in this course. From these I selected two problems about polyominoes which I was able to solve. The solution of the first problem is recently submitted [93] and the second problem is the topic of this master thesis.

Beside proving a few theorems about maximum convex hulls of polyominoes we are interested in giving the exact numbers of particular kinds of polyominoes in the appendix (for small parameters). And I would like to give an overview of the literature about enumerating polyominoes in the bibliography.

Before thanking all the people who helped me in writing this thesis I would like to give the reader a very important hint. In my definitions and proofs I preferred to take a very compact form. Therefore I would like to ask the reader to be patient in reading this thesis to get the right impression.

Acknowledgments

I would like to thank Prof. Dr. Adalbert Kerber and Prof. Dr. Heiko Harborth for looking after this thesis. For reading the manuscript I am thankful to Sonja Ertel, Christel Jantos, Nina Jantos, Heike Kurz, Frank Liczba, Armin Rund, Tobias Schneider, Steffi Sutter, Stefan Tuffner, and Nicole Willert. I am especially thankful to Matthias Koch who read the manuscript very carefully, doubt almost all of my considerations, and convinced me with great perseverance of my errors.

Declaration

This is to certify that I wrote this thesis on my own and that the references include all the sources of information I have utilized. This thesis is freely available for study purposes.

Bayreuth, 25. March 2004 ....

Chapter 1

Introduction

A **polyomino** is a connected interior-disjoint union of axis-aligned unit squares joined edge-to-edge. In other words, it is an edge-connected union of cells in the planar square lattice. There are at least three ways to define two polyominoes as equivalent, namely factoring out just translations (fixed polyominoes), rotations and translations (chiral polyominoes), or reflections, rotations and translations (free polyominoes). In the literature polyominoes are sometimes named **animals** or one speaks of the **cell-growth problem** [89, 112]. For the origin of polyominoes we quote Klarner [90]: "Polyominoes have a long history, going back to the start of the 20th century, but they were popularized in the present era initially by Solomon Golomb [66-73], then by Martin Gardner in his *Scientific American* columns." To give an illustration of polyominoes Figure 1.1 depicts the free polyominoes consisting of at most 5 unit squares.

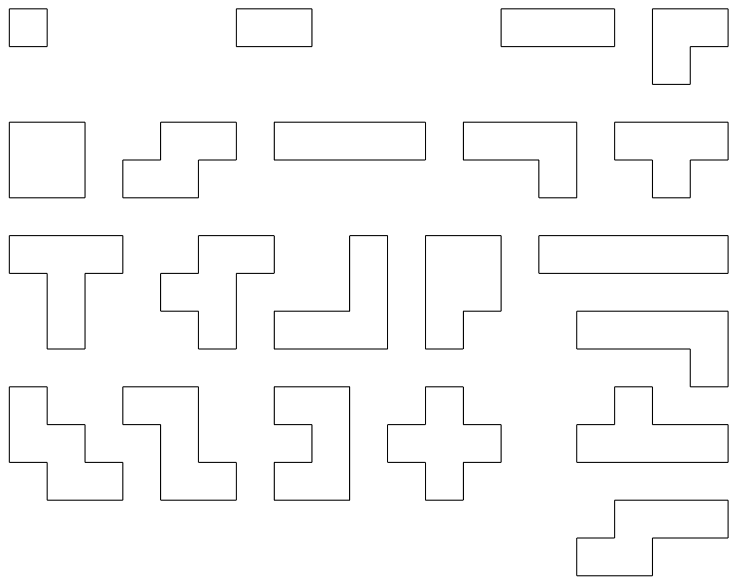

Figure 1.1. Polyominoes with at most 5 squares.

There are several generalizations of polyominoes i.e. polyiamonds (edge-to-edge unions of unit equilateral triangles) [13, 64, 78, 104, 120], polyhexes (edge-to-edge unions of unit regular hexagons) [11, 63, 64, 104], polyabolos (edge-to-edge unions of unit right isosceles triangles) [63], polycubes (face-to-face unions of unit cubes) [3, 105], etc. One can also define polyominoes as connected systems of cells on Archimedean tessellations [14]. In this thesis we regard a d-dimensional polyomino as a facet-to-facet connected system of d-dimensional unit hypercubes. If nothing else is mentioned the term polyomino is used for free polyomino.

Before we introduce the theorems of this thesis we would like to mention a few applications and problems for polyominoes. The term cell-growth problem certainly suggests applications in medicine and biology. Polyominoes are useful for the *Ising Model* [32] modeling neural networks, flocking birds, beating heart cells, atoms, protein folds, biological membrane, social behavior, etc. Further applications of polyominoes are in the fields of chemistry and physics. As problems concerning polyominoes one might mention counting polyominoes [1,2,4,5,10,15-31,34-38,40-62,65,74,76,79,82-89,92,94,96-115,119,121-124,126], generating polyominoes [83, 85], achievement games [11, 12, 13, 14, 78, 80] and extremal animals [8, 77, 81, 93]. In Appendix A we give tables for the exact number of some types of polyominoes for small numbers of cells.

This thesis is about polyominoes with maximum convex hull. At the end of this introduction we would like to mention the proven theorems.

In [8] it is proven by Bezdek, Braß, and Harborth that the area of the convex hull of any facet-to-facet connected system of n unit squares is at most $n + \frac{1}{2}\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor$. We will prove their conjecture for the d-dimensional case.

Theorem 1. The d-dimensional volume of the convex hull of any facet-to-facet connected system of n unit hypercubes is at most

$$\sum_{I\subseteq\{1,\ldots,d\}} \frac{1}{|I|!} \prod_{i\in I} \left\lfloor \frac{n-2+i}{d} \right\rfloor .$$

The authors of [8] also asked for the number of different polyominoes with n cells and maximum area of the convex hull. We enumerated them for the $\mathbb{R}^2$.

Theorem 2. The number $c_2(n)$ of polyominoes in $\mathbb{R}^2$ with maximum area of the convex hull is given by

$$n \equiv 0 \pmod 4 : c_2(n) = \frac{n^3 - 2n^2 + 4n}{16},$$

$$n \equiv 1 \pmod 4 : c_2(n) = \frac{n^3 - 2n^2 + 13n + 20}{32},$$

$$n \equiv 2 \pmod 4 : c_2(n) = \frac{n^3 - 2n^2 + 4n + 8}{16},$$

$$n \equiv 3 \pmod 4 : c_2(n) = \frac{n^3 - 2n^2 + 5n + 8}{32}.$$

Knowing the maximum area of the convex hull, one can also ask for which numbers a there is a polyomino with n cells and an area a of the convex hull. For the 2-dimensional case the situation is described by the following statement.

Theorem 3. *The existence of a 2-dimensional polyomino consisting of n cells with an area a of the convex hull is equivalent to $a \in A_n$ where*

$$A_n = \begin{cases} \left\{ n + \frac{m}{2} \middle| m \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, m \in \mathbb{N}_0 - \{1\} \right\} & : if\, n+1\, is\, prime, \\ \left\{ n + \frac{m}{2} \middle| m \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, m \in \mathbb{N}_0 \right\} & : else\,. \end{cases}$$

Chapter 2

Proof of Theorem 1

We will first prove the theorem for $d = 2, 3$ before we will prove it in any dimension.

Definition 2.1.

$$f_2(l_1, l_2, v_1, v_2) = 1 + (l_1 - 1) + (l_2 - 1) + \frac{(l_1-1)(l_2-1)}{2} + \\ v_1 + v_2 + \frac{v_1(l_2-1)}{2} + \frac{v_2(l_1-1)}{2} + \frac{v_1 v_2}{2}.$$

The standard coordinate axes of $\mathbb{R}^d$ are numbered $1, \ldots, d$. Every d-dimensional polyomino has the smallest surrounding box with side length $l_1, \ldots, l_d$, where l_i is the length in direction i. If we build up a polyomino cell by cell then after adding a cell one of the l_i will increase by 1 or none of the l_i will increase. In the second case we increase v_i by one, where the new hypercube has a facet-neighbor in direction of axis i. If N is the set of axis-directions of facet-neighbors of the new hypercube, then v_i will increase by one for only one $i \in N$. Since at this position there is the possibility to choose, we must face the fact that there might be different tuples $(l_1, \ldots, l_d, v_1, \ldots, v_d)$ for one polyomino. We define $v_1 = \ldots = v_d = 0$ for the polyomino consisting of a single hypercube. This definition of l_i and v_i leads to the following equation

$$n = 1 + \sum_{i=1}^{d}(l_i - 1) + \sum_{i=1}^{d} v_i\,. \qquad (\dagger)$$

Lemma 2.2. The area of the convex hull of a 2-dimensional polyomino is at most $f_2(l_1, l_2, v_1, v_2)$.

Proof. We prove the statement by induction on n, using equation ($\dagger$). For $n = 1$ only $l_1 = l_2 = 1$, $v_1 = v_2 = 0$ is possible. With $f_2(1, 1, 0, 0) = 1$ the induction base is done. As induction hypothesis we assume that Lemma 2.2 is proven for all possible tuples (l_1, l_2, v_1, v_2) with $1+\sum_{i=1}^{2}(l_i-1)+\sum_{i=1}^{2} v_i = n - 1$.

Due to symmetry we consider only the growth of l_1 or v_1, and the area a of the convex hull by adding the n-th square.

(i) l_1 is increased by 1:

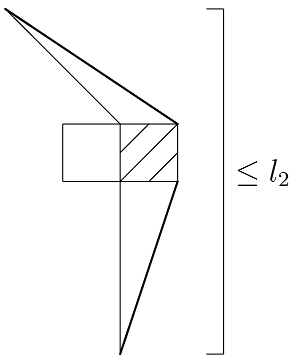

Figure 2.1. Increasing l_1.

We depict (see Figure 2.1) the new square by 3 diagonal lines. Since l_1 is increased the new square must have a left or a right neighbor. Without loss of generality it has a left neighbor. The new square contributes at most 2 (thick) lines to the convex hull of the polyomino. As we draw lines from the neighbor square to the endpoints of the new lines we see that the growth is at most $1 + \frac{l_2-1}{2}$, a growth of 1 for the new square and the rest for the triangles. Since $f_2(l_1 + 1, l_2, v_1, v_2) - f_2(l_1, l_2, v_1, v_2) = 1 + \frac{l_2-1}{2} + \frac{v_2}{2}$ the induction step follows.

(ii) v_1 is increased by one:

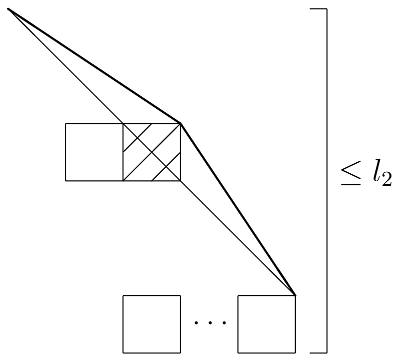

Figure 2.2. Increasing v_1.

Again we depict (Figure 2.2) the new square by 3 diagonal lines. Without loss of generality the new square has a left neighbor, and contributes at most 2 (thick) lines to the convex hull of the polyomino. As l_1 is not increased there must be a square in the same column as the new square. Similar to (i) we draw lines from the neighbor square to the endpoints of the new lines and see that the growth of the area of the convex hull is less than $\frac{l_2-1}{2}$. With $f_2(l_1, l_2, v_1 + 1, v_2) - f_2(l_1, l_2, v_1, v_2) = 1 + \frac{l_2-1}{2} + \frac{v_2}{2}$ the induction step follows.

□

Lemma 2.3. The area of the convex hull of a polyomino with n unit squares is at most $n + \frac{1}{2}\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$.

Proof.
For given n we determine the maximum of $f_2(l_1, l_2, v_1, v_2)$. We may assume $v_1 = 0$ because $f_2(l_1 + 1, l_2, v_1 - 1, v_2) - f_2(l_1, l_2, v_1, v_2) = 0$. We may also assume $v_2 = 0$ and $l_1 \leq l_2$ due to the symmetry of $f_2(l_1, l_2, v_1, v_2)$. The maximum of $f_2(l_1, l_2, 0, 0)$ cannot be attained for $l_2 - l_1 > 1$ because $f_2(l_1+1, l_2-1, 0, 0) - f_2(l_1, l_2, 0, 0) = \frac{l_2-l_1-1}{2} > 0$. Thus we conclude $l_2 - l_1 \leq 1$ and by using equation (†) we obtain $l_1 = \left\lfloor \frac{n+1}{2} \right\rfloor$, $l_2 = \left\lfloor \frac{n+2}{2} \right\rfloor$. Inserting in Lemma 2.2 yields $f_2(l_1, l_2, v_1, v_2) \leq n + \frac{1}{2}\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$. This maximum is attained for example as in Figure 2.3.

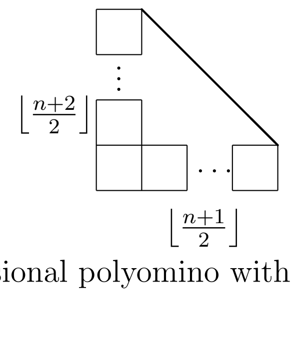

Figure 2.3. 2-dimensional polyomino with maximum convex hull.

□

Definition 2.4.

$$f_3(l_1, l_2, l_3, v_1, v_2, v_3) = 1 + (l_1 - 1) + (l_2 - 1) + (l_3 - 1) + \frac{(l_1-1)(l_2-1)}{2} + \frac{(l_1-1)(l_3-1)}{2} + \frac{(l_2-1)(l_3-1)}{2} + \frac{(l_1-1)(l_2-1)(l_3-1)}{6} + \frac{v_1(l_2-1)}{2} + \frac{v_1(l_3-1)}{2} + \frac{v_2(l_1-1)}{2} + \frac{v_2(l_3-1)}{2} + \frac{v_3(l_1-1)}{2} + \frac{v_3(l_2-1)}{2} + \frac{v_1(l_2-1)(l_3-1)}{6} + \frac{v_2(l_1-1)(l_3-1)}{6} + \frac{v_3(l_1-1)(l_2-1)}{6} + \frac{v_1 v_2(l_3-1)}{6} + \frac{v_1 v_3(l_2-1)}{6} + \frac{v_2 v_3(l_1-1)}{6} + v_1 + v_2 + v_3 + \frac{v_1 v_2}{2} + \frac{v_1 v_3}{2} + \frac{v_2 v_3}{2} + \frac{v_1 v_2 v_3}{6}.$$

Lemma 2.5. The volume of the convex hull of a 3-dimensional polyomino is at most $f_3(l_1, l_2, l_3, v_1, v_2, v_3)$.

Proof. We prove the statement by induction on n, using equation (†). For $n = 1$ only $l_1 = l_2 = l_3 = 1$, $v_1 = v_2 = v_3 = 0$ is possible. With $f_3(1, 1, 1, 0, 0, 0) = 1$ the induction base is done. As induction hypothesis we assume that Lemma 2.5 is proven for all possible tuples $(l_1, l_2, l_3, v_1, v_2, v_3)$ with $1 + \sum_{i=1}^{3}(l_i - 1) + \sum_{i=1}^{3} v_i = n - 1$.

Due to symmetry we consider only the growth of l_1 or v_1, and the volume a of the convex hull by adding the n-th cube.

(i) l_1 is increased by 1:

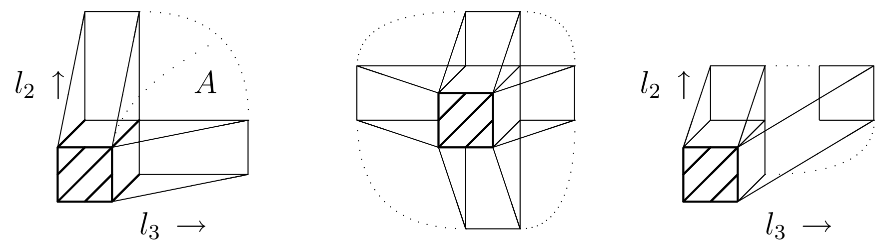

Figure 2.4. Increasing l_1 in the 3-dimensional case.

As in the proof of Lemma 2.2 we draw the lines of the convex hull of the n-th cube and its neighbor cube N, depicted in Figure 2.4. To be more precisely each line of the new convex hull has a corner point X of the upper face of the n-th cube as an endpoint. We will denote the second endpoint of this line by Y. In direction of axis 1 there is a corner point $\overline{X}$ of the bottom face of the n-th cube. Because $\overline{X}$ is also a corner point of N the line $\overline{X}Y$ must be part of the old convex hull if Y is part of the old convex hull. In this case we draw the line $\overline{X}Y$. We draw all such lines XY and $X\overline{X}$. If Y is part of the old convex hull we also draw the line $\overline{X}Y$. In the other case Y is also a corner point of the upper face of the new cube and we draw the line $\overline{XY}$ where $\overline{Y}$ is similar defined as $\overline{X}$.

Doing this we have constructed a geometrical body which contains the increase of the convex hull and which is subdivided into nice geometrical objects O_i with volume $\frac{base \times height}{k}$, $k \in \{1,2,3\}$ each. The cases $k=1$, $k=2$, or $k=3$ correspond to a box, a prism, or a tetrahedron.

We project the convex hull of the whole polyomino into the plane spanned by the vector of axis direction 2 and the vector of axis direction 3 and receive an area A. From Lemma 2.2 we know that $A \leq f_2(l_2, l_3, v_2, v_3)$ because A is the convex hull of a 2-dimensional polyomino with parameters $\overline{l_2}, \overline{l_3}, \overline{v_2}, \overline{v_3}$ where we can assume $\overline{l_2} \leq l_2$, $\overline{l_3} \leq l_3$, $\overline{v_2} \leq v_2$, and $\overline{v_3} \leq v_3$. If we apply the same projection to an O_i we get an area A_i. Due to the construction the A_i are non overlapping and we get $\sum A_i \leq A$. We use Cavalieri's theorem to determine the volume of O_i to be $\frac{A_i \times 1}{k_i}$, $k_i \in \{1,2,3\}$. More precisely, we choose lines of the form $\overline{X}X$ as height and lift the old base up until it is orthogonal to axis direction 1. Thus we may assign a factor $\frac{1}{k}$ to each piece of the area A to bound the growth of the volume of the convex hull.

Now we consider the upper face (depicted by 3 diagonal lines) of the new cube. It consists of one area, 4 edges, and 4 corner points. This face is parallel to the area A, so it contributes a volume of 1 to the convex hull. (This is exactly the n-th cube.) Now consider the two edges orthogonal to axis-direction 2. By a look at the middle picture of Figure 2.4 we notice that we have a maximal contribution of $\frac{1 \times 1 \times (l_2 - 1)}{2}$ to the volume of the convex hull for these two edges. Analog in direction 3 we have a maximal contribution of $\frac{1 \times 1 \times (l_3 - 1)}{2}$. Now there are only the 4 corner points and the dotted part of the area A left. So we have a maximal contribution of $\frac{f_2(l_2, l_3, v_2, v_3) - (l_2 - 1) - (l_3 - 1) - 1}{3}$ to the volume of the convex hull. Summing up all contributions including the n-th cube we get $1 + \frac{l_2-1}{2} + \frac{l_3-1}{2} + \frac{(l_2-1)(l_3-1)}{6} + \frac{v_2(l_3-1)}{6} + \frac{v_3(l_2-1)}{6} + \frac{v_2}{3} + \frac{v_3}{3} + \frac{v_2 v_3}{6}$. Now consider $f_3(l_1+1, l_2, l_3, v_1, v_2, v_3) - f_3(l_1, l_2, l_3, v_1, v_2, v_3) = 1 + \frac{l_2-1}{2} + \frac{l_3-1}{2} + \frac{(l_2-1)(l_3-1)}{6} + \frac{v_2(l_3-1)}{6} + \frac{v_3(l_2-1)}{6} + \frac{v_2}{2} + \frac{v_3}{2} + \frac{v_2 v_3}{6}$. Since this difference is not less than the maximal contribution of the new cube we get the induction step.

One should spend a little thought on the non-deterministic definition of the

v_i. We need the v_i for estimating the area A. If a cube, with neighbors in 2 or 3 directions is added then there is a choice of that v_i being increased by one. If the 2 directions are directions 2 and 3, then the estimation for A is valid as we have seen in the proof of Lemma 2.2. If there is a neighbor in direction 1 then the projected area A is not increased. Thus we have seen that the estimations is valid for any choice of the v_i.

(ii) v_1 is increased by one:
Similar to case (i) we choose the same decomposition of the increase of the convex hull and assign a factor $\frac{1}{k}$ to each piece of the area A. The factor $\frac{1}{1}$ can be assigned only to a piece of area at most 1. It does not harm our estimations if the real factor is $\frac{1}{2}$ or $\frac{1}{3}$ since $\frac{1}{1} > \frac{1}{2} > \frac{1}{3}$. Now we consider the volume of the possible prisms. For a prism we need a complete face of the new cube. So we can use at most $l_2 + l_3 - 1$ of A for the prisms. As we already have assigned the factor $\frac{1}{1}$ to a piece of area 1 there is only $l_2 - 1 + l_3 - 1$ left. Thus we get the same estimation as in (i). Since a part of the new cube is already part of the old convex hull the volume of the convex hull of a polyomino with parameters $l_1, l_2, l_3, v_1, v_2, v_3$ is strictly less than $f_3(l_1, l_2, l_3, v_1, v_2, v_3)$.

□

Lemma 2.6. The volume of the convex hull of a polyomino consisting of n unit cubes is at most

$$1 + \left\lfloor \frac{n-1}{3} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor + \frac{\left\lfloor \frac{n-1}{3} \right\rfloor \left\lfloor \frac{n}{3} \right\rfloor}{2} + \frac{\left\lfloor \frac{n-1}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor}{2} + \frac{\left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor}{2} + \frac{\left\lfloor \frac{n-1}{3} \right\rfloor \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor}{6}.$$

Proof.
For given n we determine the maximum of $f_3(l_1, l_2, l_3, v_1, v_2, v_3)$. We may assume $v_1 = 0$ because $f_3(l_1 + 1, l_2, l_3, v_1 - 1, v_2, v_3) - f_3(l_1, l_2, l_3, v_1, v_2, v_3) = 0$. Due to the symmetry of $f_3(l_1, l_2, l_3, v_1, v_2, v_3)$ we may also assume $v_2 = v_3 = 0$ and $l_1 \leq l_2 \leq l_3$. The maximum of $f_3(l_1, l_2, l_3, 0, 0, 0)$ cannot be attained for $l_3 - l_1 > 1$ because $f_3(l_1 + 1, l_2, l_3 - 1, 0, 0, 0) - f_3(l_1, l_2, l_3, 0, 0, 0) = \frac{(l_3 - l_1 - 1)(l_2 + 2)}{6} > 0$. Thus we conclude $l_3 - l_1 \leq 1$ and by using equation (†) we receive $l_i = \left\lfloor \frac{n+1+i}{3} \right\rfloor$. By inserting this term in Lemma 2.5 we get

the desired estimation. An example of an extremal configuration consist of 3 pairwise orthogonal linear arms of $\left\lfloor \frac{n-2+i}{3} \right\rfloor$ cubes ($i = 1 \ldots 3$) joined at a central cube.

$\square$

In the d-dimensional case we use the same structure of the lemmas in the 2- and 3-dimensional case. At first we generalize Definition 2.1 and Definition 2.4.

Definition 2.7.

$$f_d(l_1, \ldots, l_d, v_1, \ldots, v_d) = \sum_{I \subseteq \{1,\ldots,d\}} \frac{1}{|I|! 2^{d-|I|}} \sum_{b=0}^{2^d-1} \prod_{i \in I} q_{b,i}$$

with $d \geq 1$ and $b = \sum\limits_{j=1}^{d} b_j 2^{j-1}$, $b_j \in \{0, 1\}$, $q_{b,i} = \begin{cases} l_i - 1 & for \quad b_i = 0\,, \\ v_i & for \quad b_i = 1\,. \end{cases}$

Lemma 2.8. The d-dimensional volume of the convex hull of a polyomino consisting of n unit hypercubes is at most $f_d(l_1, \ldots, l_d, v_1, \ldots, v_d)$.
Proof. We prove the statement by double induction on d and n, using equation (†). The cases $d = 2, 3$ are already treated, so we may assume that the lemma is proven for the $\overline{d}$-dimensional volume with $\overline{d} < d$. For $n = 1$ only $l_i = 1$, $v_i = 0$ $i \in \{1, \ldots, d\}$ is possible. With $f_d(1, \ldots, 1, 0, \ldots, 0) = 1$ the induction base for n is done. As induction hypothesis we assume that the lemma is proven for all possible tuples $(l_1, \ldots, l_d, v_1, \ldots, v_d)$ with $1 + \sum_{i=1}^{d}(l_i - 1) + \sum_{i=1}^{d} v_i = n - 1$.
Due to symmetry we consider only the growth of l_1 or v_1, and the volume a of the convex hull by adding the n-th hypercube.

(i) l_1 is increased by one:
We use the same construction as in the proof of Lemma 2.5 to obtain a geometric objects O_i as estimation for the increase of the convex hull. The projection of the convex hull of the polyomino into the hyperplane orthogonal to axis direction 1 yields a volume A which is the convex hull of a $(d-1)$-dimensional polyomino with parameters $\overline{l_2}, \ldots, \overline{l_d}, \overline{v_2}, \ldots, \overline{v_d}$ where we can assume $\overline{l_2} \le l_2, \ldots, \overline{l_d} \le l_d$ and $\overline{v_2} \le v_2, \ldots, \overline{v_d} \le v_d$. From the induction hypothesis we know that $A \le f_{d-1}(l_2, \ldots, l_d, v_2, \ldots, v_d)$. As in the proof of Lemma 2.5 we also project the O_i yielding volumes A_i and apply Cavalieri's theorem to determine the volume of O_i to be $\frac{A_i \times 1}{k_i}$, $k_i \in \{1, \ldots, d\}$. Because the A_i are non overlapping we may split A into different parts multiplied by $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{d}$, respectively, to get the maximum growth of the volume by adding the n-th cube. We choose the parts in a way that the parts with the higher factors are as big as theoretical possible.

For every $0 \le r \le d-1$ we can consider the sets $\{i_1, i_2, \ldots, i_r\}$ with $1 \ne i_a \ne i_b$ for $a \ne b$. Let Y be such a set. Define $\overline{Y} = \{j_1, \ldots, j_{d-r-1}\}$ by $Y \cap \overline{Y} = \{\}$ and $Y \cup \overline{Y} = \{2, \ldots, d\}$. So the vector space spanned by the axis directions of Y and the vector space spanned by the axis directions of $\overline{Y}$ are orthogonal. In the proof of Lemma 2.5 we started our consideration with the upper face of the new cube. This would correspond to $Y = \{2, 3\}$ since this face is parallel to the vector space spanned by axis direction 2 and axis direction 3. The two edges parallel to the axis direction 3 would be described by $Y = \{3\}$ and $\overline{Y} = \{2\}$. If we project the convex hull in the vector space spanned by $\overline{Y}$ the resulting volume is at most $f_{d-r-1}(l_{j_1}, \ldots, l_{j_{d-r-1}}, v_{j_1}, \ldots, v_{j_{d-r-1}})$ since it is the convex hull of a $(d-r-1)$-dimensional polyomino. For our last example this would be $f_1(l_2, v_2) = 1 + (l_2 - 1) + v_2$. Since $\overline{Y}$ has cardinality $d-r-1$ the set Y yields a contribution of $\frac{1}{d-r} f_{d-r-1}(l_{j_1}, \ldots, l_{j_{d-r-1}}, v_{j_1}, \ldots, v_{j_{d-r-1}})$ to the volume of the convex hull. In terms of Definition 2.7 this is

$$\frac{1}{d-r} \sum_{I \subseteq \{j_1, \ldots, j_{r-d-1}\}} \frac{1}{|I|! 2^{d-r-1-|I|}} \sum_{b=0}^{2^{d-r-1}-1} \prod_{i \in I} q_{b,i} \, .$$

Our aim was to assign the maximum possible factor to each part of A. For that reason we count for Y a maximum contribution of

$$\frac{1}{d-r} \frac{1}{|d-r-1|!} \sum_{b=0}^{2^{d-r-1}-1} \prod_{i \in \overline{Y}} q_{b,i}$$

to the volume of the convex hull.

If we do so for all possible sets Y we have assigned a factor between 1 and $\frac{1}{d}$ to every summand of $f_{d-1}(l_2, \ldots, l_d, v_2, \ldots, v_d)$. To get the induction step now we have to remark that the above described sum with its factors is exactly the difference between $f_d(l_1+1, \ldots, l_d, v_1, \ldots, v_d)$ and $f_d(l_1, \ldots, l_d, v_1, \ldots, v_d)$.

(ii) v_1 is increased by one:
Due to the symmetry of l_i and v_i in Definition 2.7 this is analog to (i). Additionally we remark that the maximum cannot be achieved in this case since there is already a cube on this level. Therefore we double count a part of the contribution of the new cube to the volume of the convex hull in our estimations.

□

Theorem 1. The d-dimensional volume of the convex hull of any facet-to-facet connected system of n unit hypercubes is at most

$$\sum_{I \subseteq \{1,\ldots,d\}} \frac{1}{|I|!} \prod_{i \in I} \left\lfloor \frac{n-2+i}{d} \right\rfloor .$$

Proof.

For given n we determine the maximum of $f_d(l_1, \ldots, f_d, v_1, \ldots, v_d)$. Because of $f_d(l_1+1, l_2, \ldots, l_d, v_1-1, v_2, \ldots, v_d) - f_d(l_1, l_2, \ldots, l_d, v_1, v_2, \ldots, v_d) = 0$ we may assume $v_1 = 0$. The symmetry of $f_d(l_1, \ldots, l_d, v_1, \ldots, l_d)$ allows us to assume $v_2 = \ldots = v_d = 0$ and $l_1 \leq l_2 \leq \ldots \leq l_d$. The maximum of $f_d(l_1, \ldots, l_d, 0, \ldots, 0)$ cannot be attained for $l_d - l_1 > 1$ because of $f_d(l_1+1, l_2, \ldots, l_{d-1}, l_d-1, 0, 0, \ldots, 0) - f_d(l_1, l_2, \ldots, l_d, 0, 0, \ldots, 0) > 0$. We obtain this inequality by the following consideration. If a summand of $f_d(\ldots)$ contains the term l_1 and does not contain l_d then there will be a corresponding summand with l_1 replaced by l_d, so those terms equalize each other in the above difference. Clearly the summands containing none of the terms l_1 or l_d equalize each other in the difference. So there are left only the summands with both terms l_1 and l_d. Since $(l_1+1-1)(l_d-1-1) - (l_1-1)(l_d-1) = l_d - l_1 - 1 > 0$ the above inequality is fulfilled. Thus we conclude $l_d - l_1 \leq 1$ and by using equation (†) we get $l_i = \left\lfloor \frac{n-2+i+d}{d} \right\rfloor$. Inserting this in Lemma 2.8 yields the desired estimation. An example of an extremal configuration consists of d pairwise orthogonal linear arms of $\left\lfloor \frac{n-2+i}{d} \right\rfloor$ cubes $(i = 1 \ldots d)$ joined to a central cube.

$\square$

Chapter 3

Proof of Theorem 2

Since we want to count all 2-dimensional polyominoes with maximum area of the convex hull we describe a bigger class of extremal polyominoes.

Lemma 3.1. Every 2-dimensional polyomino with parameters l_1, l_2, v_1, v_2 and with the maximum area of the convex hull consists of a linear strip with at most one orthogonal linear strip on each side. This polyomino fulfills $v_1 = v_2 = 0$ and the area of the convex hull is given by $f_2(l_1, l_2, 0, 0)$.

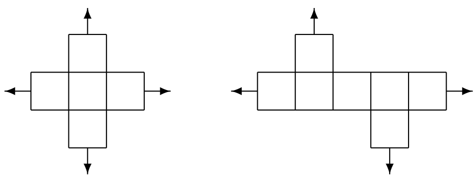

Figure 3.1. The 2 shapes of polyominoes with maximum convex hull.

Proof.
From the proof of Lemma 2.2 (ii) we conclude that $v_1 = v_2 = 0$ for a polyomino with maximum area of the convex hull. We can also conclude that every polyomino which is a part of a bigger polyomino with maximum area of the convex hull must also have the maximum possible area of the convex hull. We determine the area of the convex hull of a polyomino in the

shape of those in Figure 3.2 to be less than $f_2(l_1, l_2, 0, 0)$ thus it cannot be contained in a polyomino with maximum area of the convex hull. The conditions $v_1 = v_2 = 0$ and the forbidden sub-polyominoes of Figure 3.2 restrict the class of polyominoes to those described in Lemma 3.1. Additionally their area of the convex hull is $f_2(l_1, l_2, 0, 0)$.

□

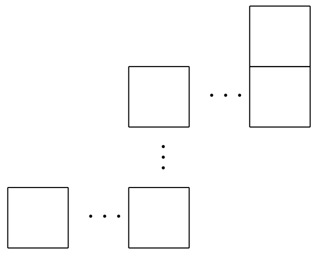

Figure 3.2. Forbidden sub-polyomino.

Now we start to count all polyominoes described by Lemma 3.1 for fixed l_1, l_2. We denote the linear strip by R and the possible two orthogonal linear strips by T_1 and T_2. Since the smallest surrounding rectangle of such a polyomino consists of l_2 rows and l_1 columns we can label the rows from 1 to l_2 and label the columns from 1 to l_1. Let r denote the row number of R if it is horizontal or the column number if R is vertical. Similar t_1 and t_2 denote the column number or row number of T_1 and T_2.

There are two trivial cases for $l_1 = 1$ or $l_2 = 1$. In all other cases we can assume that T_1 exists. We distinguish the following two cases.
shape 1: T_2 does not exist or $t_1 = t_2$.
shape 2: $t_1 \neq t_2$.
Thus a polyomino of shape 1 is described by a pair (r, t_1) and a polyomino of shape 2 by a triple (r, t_1, t_2). Since we consider with free polyominoes only we have to factor out rotations and reflections. For $l_1 \neq l_2$ it suffices to consider only the vertical and the horizontal symmetry axis of the smallest surrounding rectangle of the polyomino. If $l_1 = l_2$ we have to consider a vertical, a horizontal, and a diagonal symmetry axis. Now we distinguish whether $l_1 = l_2$ or not and the two cases for the shapes.

(i) $l_1 = l_2$, shape 1.
Due to possible reflection at the horizontal and vertical symmetry axis we can assume $1 \le r \le \lceil \frac{l_1}{2} \rceil$, $1 \le t_1 \le \lceil \frac{l_1}{2} \rceil$. Due to the diagonal symmetry axis we can assume $t_1 \le r$. As there are no more symmetries we obtain for the number of extremal polyominoes

$$c_{2,1}(l_1, l_1) = \sum_{r=1}^{\left\lceil \frac{l_1}{2} \right\rceil} \sum_{t_1=1}^{r} 1 = \sum_{r=1}^{\left\lceil \frac{l_1}{2} \right\rceil} r = \frac{\left\lceil \frac{l_1}{2} \right\rceil \left\lceil \frac{l_1+2}{2} \right\rceil}{2} .$$

(ii) $l_1 \ne l_2$, shape 1.
Due to the reflections at the horizontal and vertical symmetry axis we can assume $1 \le r \le \left\lceil \frac{l_1}{2} \right\rceil$, $1 \le t_1 \le \left\lceil \frac{l_2}{2} \right\rceil$. Thus we get

$$c_{2,2}(l_1, l_2) = \left\lceil \frac{l_1}{2} \right\rceil \left\lceil \frac{l_2}{2} \right\rceil .$$

(iii) $l_1 = l_2$, shape 2.
Due to the diagonal symmetry axis we can assume that R is a vertical strip. The existence of T_2 and the vertical symmetry axis let us assume $2 \le r \le \lceil \frac{l_1}{2} \rceil$. Now we consider the two cases $2 \le r \le \lfloor \frac{l_1}{2} \rfloor$ and $\lfloor \frac{l_1}{2} \rfloor < r \le \lceil \frac{l_1}{2} \rceil$.

(a) $2 \le r \le \lfloor \frac{l_1}{2} \rfloor$.
In this case we must only consider the horizontal symmetry axis. Which yields $1 \le t_1 \le \lceil \frac{l_1}{2} \rceil$. For $1 \le t_1 \le \lfloor \frac{l_1}{2} \rfloor$ there are no more symmetry axes to consider and we get $\left\lfloor \frac{l_1-2}{2} \right\rfloor \left\lfloor \frac{l_1}{2} \right\rfloor (l_1 - 1)$ possibilities. If $\lfloor \frac{l_1}{2} \rfloor < t_1 \le \lceil \frac{l_1}{2} \rceil$ the horizontal symmetry axis of the smallest surrounding rectangle forces $t_2 < \lceil \frac{l_1}{2} \rceil$. This results in $\left\lfloor \frac{l_1-2}{2} \right\rfloor \left\lfloor \frac{l_1}{2} \right\rfloor$ possibilities. Thus we get

$$c_{2,3,1}(l_1, l_1) = \left\lfloor \frac{l_1 - 2}{2} \right\rfloor \frac{l_1(l_1 - 1)}{2} .$$

(b) $\lfloor \frac{l_1}{2} \rfloor < r \leq \lceil \frac{l_1}{2} \rceil$.
Considering the vertical symmetry axis may us assume that t_1 lies nearer at the border: $\min(t_1, l_1+1-t_1) \leq \min(t_2, l_1+1-t_2)$. The horizontal symmetry axis yields $1 \leq t_1 \leq \lfloor \frac{l_1}{2} \rfloor$, so we get

$$c_{2,3,2}(l_1, l_1) = \left(\left\lceil \frac{l_1}{2} \right\rceil - \left\lfloor \frac{l_1}{2} \right\rfloor \right) \sum_{t_1=1}^{\lfloor \frac{l_1}{2} \rfloor} \sum_{t_2=t_1+1}^{l_1+1-t_1} 1 = \left(\left\lceil \frac{l_1}{2} \right\rceil - \left\lfloor \frac{l_1}{2} \right\rfloor \right) \sum_{t_1=1}^{\lfloor \frac{l_1}{2} \rfloor} l_1 + 1 - 2t_1$$

$$= \left(\left\lceil \frac{l_1}{2} \right\rceil - \left\lfloor \frac{l_1}{2} \right\rfloor \right) ((l_1+1) \left\lfloor \frac{l_1}{2} \right\rfloor - \left\lfloor \frac{l_1}{2} \right\rfloor \left\lfloor \frac{l_1+2}{2} \right\rfloor) = \left(\left\lceil \frac{l_1}{2} \right\rceil - \left\lfloor \frac{l_1}{2} \right\rfloor \right) \left\lfloor \frac{l_1}{2} \right\rfloor \left\lceil \frac{l_1}{2} \right\rceil .$$

(iv) $l_1 \neq l_2$, shape 2, R is a vertical strip.
The existence of T_2 and the vertical symmetry axis let us assume $2 \leq r \leq \lceil \frac{l_1}{2} \rceil$. Now we consider the two cases $2 \leq r \leq \lfloor \frac{l_1}{2} \rfloor$ and $\lfloor \frac{l_1}{2} \rfloor < r \leq \lceil \frac{l_1}{2} \rceil$.

(a) $2 \leq r \leq \lfloor \frac{l_1}{2} \rfloor$.
In this case we must only consider the horizontal symmetry axis. Which yields $1 \leq t_1 \leq \lceil \frac{l_2}{2} \rceil$. For $1 \leq t_1 \leq \lfloor \frac{l_2}{2} \rfloor$ there are no more symmetry axes to consider and we get $\left\lfloor \frac{l_1-2}{2} \right\rfloor \left\lfloor \frac{l_2}{2} \right\rfloor (l_2 - 1)$ possibilities. If $\lfloor \frac{l_2}{2} \rfloor < t_1 \leq \lceil \frac{l_2}{2} \rceil$ the horizontal symmetry axis of the smallest surrounding rectangle forces $t_2 < \lceil \frac{2_1}{2} \rceil$. This results in $\left\lfloor \frac{l_2-2}{2} \right\rfloor \left\lfloor \frac{l_2}{2} \right\rfloor$ possibilities. Thus we get

$$c_{2,4,1}(l_1, l_2) = \left\lfloor \frac{l_1 - 2}{2} \right\rfloor \frac{l_2(l_2 - 1)}{2} .$$

(b) $\lfloor \frac{l_1}{2} \rfloor < r \leq \lceil \frac{l_1}{2} \rceil$.
Considering the vertical symmetry axis may us assume that t_1 lies nearer at the border: $\min(t_1, l_1+1-t_1) \leq \min(t_2, l_1+1-t_2)$. The horizontal symmetry axis yields $1 \leq t_1 \leq \lfloor \frac{l_2}{2} \rfloor$, so we receive similar to (iii)(b)

$$c_{2,4,2}(l_1, l_2) = \left(\left\lceil \frac{l_1}{2} \right\rceil - \left\lfloor \frac{l_1}{2} \right\rfloor \right) \left\lfloor \frac{l_2}{2} \right\rfloor \left\lceil \frac{l_2}{2} \right\rceil .$$

(v) $l_1 \neq l_2$, shape 2, R is a horizontal strip.
If we interchange l_1 and l_2 we can conclude from (iv)

$$c_{2,5,1}(l_1, l_2) = \left\lfloor \frac{l_2 - 2}{2} \right\rfloor \frac{l_1(l_1 - 1)}{2}$$

and

$$c_{2,5,2}(l_1, l_2) = \left(\left\lceil \frac{l_2}{2} \right\rceil - \left\lfloor \frac{l_2}{2} \right\rfloor \right) \left\lfloor \frac{l_1}{2} \right\rfloor \left\lceil \frac{l_1}{2} \right\rceil .$$

We summarize the results including the trivial cases.

$l_1 = l_2$:

$l_1 \equiv 0 \ \ (mod\ 2)$: $c_2(l_1, l_1) = \frac{2l_1^3 - 5l_1^2 + 6l_1}{8}$ for $l_1 \geq 0$,
$l_1 \equiv 1 \ \ (mod\ 2)$: $c_2(l_1, l_1) = \frac{2l_1^3 - 5l_1^2 + 10l_1 + 1}{8}$ for $l_1 \geq 1$.

$l_1 \neq l_2$:

$l_1 \equiv 0, l_2 \equiv 0 \ \ (mod\ 2)$: $c_2(l_1, l_2) = \frac{l_1 l_2 (l_1 + l_2 - 1) - 2l_1^2 - 2l_2^2 + 2l_1 + 2l_2}{4}$ for $l_1, l_2 \geq 2$,

$$l_1 \equiv 1, l_2 \equiv 0 \ \ (mod\ 2):\ c_2(l_1, l_2) = \begin{cases} \frac{l_1 l_2 (l_1 + l_2 - 1) - 2l_1^2 - 2l_2^2 + 4l_2 + 2l_1}{4} & \text{for } l_1 \geq 3, l_2 \geq 2\,, \\ 1 & \text{for } l_1 = 1, l_2 \geq 2\,, \end{cases}$$

$$l_1 \equiv 0, l_2 \equiv 1 \ \ (mod\ 2):\ c_2(l_1, l_2) = \begin{cases} \frac{l_1 l_2 (l_1 + l_2 - 1) - 2l_1^2 - 2l_2^2 + 4l_1 + 2l_2}{4} & \text{for } l_1 \geq 2, l_2 \geq 3\,, \\ 1 & \text{for } l_1 \geq 2, l_2 = 1\,, \end{cases}$$

$$l_1 \equiv 1, l_2 \equiv 1 \ \ (mod\ 2):\ c_2(l_1, l_2) = \begin{cases} \frac{l_1 l_2 (l_1 + l_2 - 1) - 2l_1^2 - 2l_2^2 + 4l_1 + 4l_2 - 1}{4} & \text{for } l_1, l_2 \geq 3\,, \\ 1 & \text{for } else\,. \end{cases}$$

Theorem 2. The number $c_2(n)$ of polyominoes in $\mathbb{R}^2$ with maximum area of the convex hull is given by

$$n \equiv 0 \ \ (mod\ 4) : c_2(n) = \frac{n^3 - 2n^2 + 4n}{16},$$

$$n \equiv 1 \ (mod\ 4) : c_2(n) = \frac{n^3 - 2n^2 + 13n + 20}{32},$$

$$n \equiv 2 \ (mod\ 4) : c_2(n) = \frac{n^3 - 2n^2 + 4n + 8}{16},$$

$$n \equiv 3 \ \ (mod\ 4) : c_2(n) = \frac{n^3 - 2n^2 + 5n + 8}{32}.$$

Proof. We may assume $l_1 = \left\lfloor \frac{n+1}{2} \right\rfloor$ and $l_2 = \left\lfloor \frac{n+2}{2} \right\rfloor$ so Theorem 2 follows from the above.

□

Conclusion 3.2. The ordinary generating function for the number $c_2(n)$ of polyominoes in $\mathbb{R}^2$ with maximum area of the convex hull is given by

$$\frac{1 + x - x^2 - x^3 + 2x^5 + 8x^6 + 2x^7 + 4x^8 + 2x^9 - x^{10} + x^{12}}{(1 - x^2)^2(1 - x^4)^2}.$$

Chapter 4

Proof of Theorem 3

Theorem 3. The existence of a 2-dimensional polyomino consisting of n cells with area a of the convex hull is equivalent to $a \in A_n$ with

$$A_n = \begin{cases} \left\{ n + \frac{m}{2} \middle| m \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, m \in \mathbb{N}_0 - \{1\} \right\} & : \, if \, n+1 \, is \, prime, \\ \left\{ n + \frac{m}{2} \middle| m \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, m \in \mathbb{N}_0 \right\} & : \, else \, . \end{cases}$$

Proof. $\subseteq$:
Since the corner points of a polyomino are on a unit square grid the area of the convex hull must be an integral multiple of $\frac{1}{2}$. With Lemma 2.3 and the fact that the area of n unit squares is n we get that the set of possible areas of the convex hull of polyominoes with n cells must be a subset of

$$\left\{ n + \frac{m}{2} \middle| m \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, m \in \mathbb{N}_0 \right\}.$$

A polyomino consisting of n cells with area n of the convex hull must be convex. If the area of the convex hull is $n + \frac{1}{2}$ there must be a triangle of area $\frac{1}{2}$. If we extend the triangle to a square we get a polyomino consisting of $n + 1$ cells.

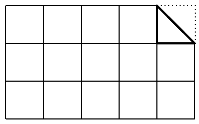

Figure 4.1. Polyominoes with n squares and area $n + \frac{1}{2}$ of the convex hull.

Thus we can construct all polyominoes consisting of n unit squares with area $n + \frac{1}{2}$ of the convex hull by deleting a square at the corner of a convex polyomino consisting of $n+1$ cells. The convex polyominoes are exactly the $a \times b$ rectangles. Thus for $n+1$ prime there is only the $1 \times (n+1)$ rectangle. Deleting a square yields area n of the convex hull.

$\supseteq$:
For $m = 0$ we have the $a \times b$ rectangles with $ab = n$ as examples. The above consideration for area $n + \frac{1}{2}$ of the convex hull yields a construction for $n+1$ composite. Now we give 6 constructions to handle the other values for m.

(i)

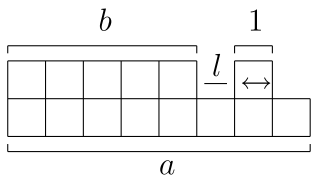

Figure 4.2. Construction 1.

We let a run from $\lceil \frac{n}{2} \rceil$ to $n-2$ and let l run from 0 to $a-b-1$. With $a+b+1 = n$ we get $a \geq b+1$ so that the construction depicted in Figure 4.2 is possible. For $n \equiv 0$ $(mod\ 2)$ the attained values for m are $(0), (2, \ldots, 4), (4, \ldots, 8), \ldots, (a-b-1, \ldots, 2a-2b-2), \ldots, (n-4, \ldots, 2n-8) = 0, 2, 3, \ldots, 2n-8$. For $n \equiv 1$ $(mod\ 2)$ the attained values for m are $(1, 2), (3, \ldots, 6), (5, \ldots, 10), \ldots, (a-b-1, \ldots, 2a-2b-2), \ldots, (n-4, \ldots, 2n-8) = 0, 2, 3, \ldots, 2n-8$.
So we can handle $2 \leq m \leq 2n-8$.

(ii)

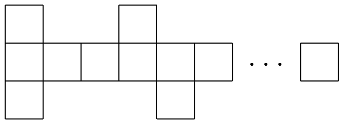

Figure 4.3. Construction 2.

For $n \geq 9$ Construction 2 yields $m = 2n - 7$. The remaining cases $5 \leq n \leq 8$ for $m = 2n - 7$ are treated in Figure 4.4.

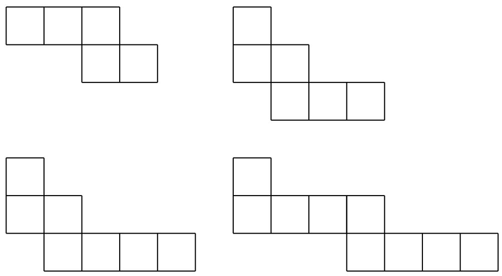

Figure 4.4. $m = 2n - 7$ for $5 \leq n \leq 8$.

(iii)

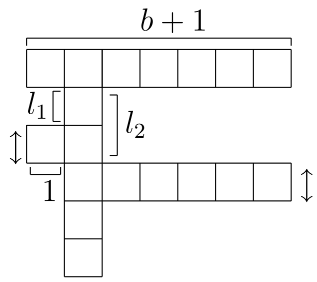

Figure 4.5. Construction 3.

The conditions for a possible construction of Figure 4.5 are $0 \leq l_1, l_2 \leq n - 2b - 2$ and $2b + 2 \leq n$. We demand $n - 2b - 2 \geq b$ which is equivalent to $b \leq \frac{n}{3}$. With given l_1, l_2, b, n it holds $m = bn - 2b^2 - 2b + l_1 + l_2(b-1)$. Because we have demanded $n - 2b - 2 \geq b - 1$ we can vary l_1 at least between 0 and $b - 2$ and so we can get by changing l_1 and l_2 all values between $b(n - 2b - 2)$

and $2b(n-2b-2)$. Now we want to combine those intervals for successive values for b. The assumption that the intervals do not intersect is equivalent to

$$\begin{aligned} & 2(b-1)(n-2(b-1)-2) < b(n-2b-2) \\ \Leftrightarrow\quad & bn-2n-2b^2+6b < 0 \\ \Leftrightarrow\quad & n(b-2) < 2b(b-3) \\ \Leftrightarrow\quad & n < 2b\tfrac{b-3}{b-2} < 2b. \end{aligned}$$

As we already have to fulfill $b \leq \frac{n}{3}$ the intervals intersect. We choose $2 \leq b \leq \left\lfloor \frac{n}{4} \right\rfloor$ and get constructions for m in the interval $2n-6, 2n-5\ldots, \left\lceil \frac{n^2-4n}{4} \right\rceil$.

(iv)

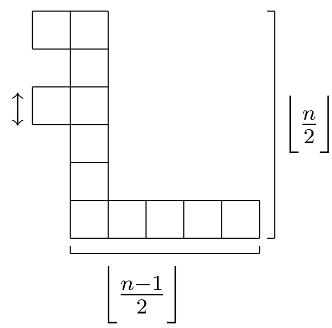

Figure 4.6. Construction 4.

For $n \geq 4$ Construction 4 is possible and for $n \equiv 0 \pmod 2$ we can get m from $\frac{n^2-4n}{4}$ to $\frac{n^2-2n-8}{4}$. For $n \equiv 1 \pmod 2$ we can get m from $\frac{n^2-4n-1}{4}$ to $\frac{n^2-2n-11}{4}$.

(v)

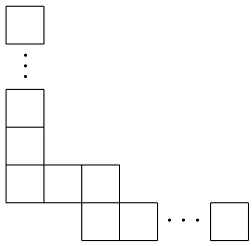

Figure 4.7. Construction 5.

The height and the width of Figure 4.7 is given by $\left\lfloor \frac{n+1}{2} \right\rfloor$ and $\left\lfloor \frac{n+2}{2} \right\rfloor$ For $n \geq 7$ Construction 5 is possible. We only need it for odd n to obtain $m = \frac{n^2-2n-7}{4}$. For $n \leq 5$ we remark $2n - 7 \geq \frac{n^2-2n-7}{4}$.

(vi)

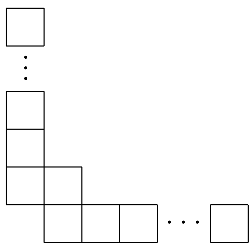

Figure 4.8. Construction 6.

The height and the width of Figure 4.8 is as in Figure 4.7. For even n we get $m = \frac{n^2-2n-4}{4}$ and for odd n we get $m = \frac{n^2-2n-3}{4}$.

The proof is completed by Figure 2.3.

$\square$

We enumerated the 2-dimensional polyominoes with maximum area of the convex hull in Theorem 2. For the minimum area n of the convex hull we enumerate them in Lemma 4.1 and for area $n + \frac{1}{2}$ of the convex hull we enumerate them in Lemma 4.2. Therefore we denote the number of divisors of an integer n by $\tau(n)$.

Lemma 4.1. The number of polyominoes consisting of n unit squares with minimum area n of the convex hull is given by $\left\lceil \frac{\tau(n)}{2} \right\rceil$.
Proof. Those polyominoes are convex (in the sense of geometry) and so they are all rectangle polyominoes. Considering the symmetries yields the division by two and the ceiling.

□

Lemma 4.2. The number of polyominoes consisting of n unit squares with area $n + \frac{1}{2}$ of the convex hull is given by $\left\lceil \frac{\tau(n+1)}{2} \right\rceil - 1$.
Proof. See Figure 4.1 and the proof of Theorem 3 for a description of those polyominoes.

□

We can also ask for an analogue version of Theorem 3 for the d-dimensional case. Because the situation in higher dimensions is more complicated we can only give partial answers.

Lemma 4.3. The volume of the convex hull of d-dimensional polyominoes consisting of n unit hypercubes is an integral multiple of $\frac{1}{d!}$.
Proof. For the determination of the convex hull of a polyomino we must only consider the set of corner points of its hypercubes. We can decompose the convex hull into smaller pieces consisting of the convex hull of $d+1$ corner points. (For $d = 2$ this would be called a triangulation.) If we denote the coordinates of the i-th corner point as $p_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,d})$ we know (for example from [91]) that the volume spanned by the points $p_1, p_2, \ldots, p_{d+1}$ is given by

$$\mathit{volume}(p_1, \ldots, p_{d+1}) = \frac{1}{d!} \begin{vmatrix} x_{1,1} & \ldots & x_{1,d} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{d+1,1} & \ldots & x_{d+1,d} & 1 \end{vmatrix}.$$

Without loss of generality we assume that the corner points of the polyomino lie on an integer grid, and so $volume(p_1, \ldots, p_{d+1})$ is an integral multiple of $\frac{1}{d!}$.

□

The combination of Lemma 4.3 and Theorem yields a weaker version of Theorem 3.

Conclusion 4.4. If there exists a d-dimensional polyomino consisting of n cells with volume v of the convex hull, then $v \in V_{d,n}$ with

$$V_{d,n} = \left\{ n + \frac{m}{d!} \middle| m \leq \sum_{I \subseteq \{1,\ldots,d\}} \frac{d!}{|I|!} \prod_{i \in I} \left\lfloor \frac{n-2+i}{d} \right\rfloor \; m \in \mathbb{N}_0 \right\} .$$

Lemma 4.5. The number of d-dimensional polyominoes consisting of n unit hypercubes with minimum volume n is the number of ways to decompose n into a set of d factors.

Proof. Those polyominoes are convex (in the sense of geometry) and so they are hyper rectangle polyominoes.

□

Chapter 5

Prospect

In the last chapters we enumerated the 2-dimensional polyominoes with maximum and the d-dimensional polyominoes with minimum volume of the convex hull. Our Theorem 1 gives the maximum possible volume of the convex hull for a d-dimensional polyomino consisting of unit hypercubes. A task for the future might be the enumeration of those d-dimensional polyominoes with maximum volume of the convex hull.

Here we do not treat the equivalent problem for polyiamonds, polyhexes or other kinds of polyominoes. For polyiamonds consisting of n unit equilateral triangles the minimum area of the convex hull is n since there are are convex (in the sense of geometry) polyiamonds for every integer n. It is not difficult to describe the shape of those extremal animals, but to find an elegant way to enumerate their number is another thing. Polyhexes with more than one cell cannot be convex (in the sense of geometry). We propose that the set of polyhexes with the minimum area of the convex hull is a subset of the set of polyhexes with minimum perimeter. The latter set is not enumerated yet, at least to the author's knowledge.

Another class of problems which is also related to our topic is the question for the maximum area of the convex hull of all edge-to-edge packings of n regular k-gons in the plane. We conjecture that for $k \geq 6$ the arrangements of n regular k-gons in the plane with maximum area of the convex hull look like the boundaries of a semicircle.

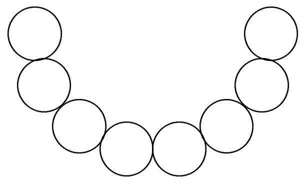

Figure 5.1. An example of circles with big area of the convex hull.

In Chapter 4 we characterized the possible values for the area of the convex hull of a 2-dimensional polyomino consisting of n unit squares. We can ask for an analogous characterization for d-dimensional polyominoes with $d > 2$.

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Appendix A

Exact numbers of polyominoes

Since I am generally interested in enumerating of polyominoes the bibliography, with most entries concerning enumeration of polyominoes, will be followed by the, at least to me, known exact numbers of some kinds of polyominoes.

When we talk about enumeration of combinatorial objects we must distinguish wether we only *count* or *construct* these objects. From a practical point of view the construction of combinatorial objects is more worthy than the pure counting. Several authors try to extend counting techniques to construction techniques. In the case of polyominoes one was only able to count polyominoes by an explicit construction of all polyominoes for a long time. The paper of I. G. Enting [51] is regarded as a breakthrough. His methods were applied and extended by several authors [123, 124]. The general method is named *Finite Lattice Method* and for polyominoes on the square lattice a specialized algorithm is called *Jensens algorithm* (also implemented by D. Knuth).

In this context I would like to mention N.J.A. Sloane's marvelous **Online Encyclopedia of Integer Sequences** [116, 117]. This archive contains over 92.000 integer sequences and numerous references. Suppose you are working on a topic where an integer sequence is involved. Calculating the first few terms and using the Look-Up interface of the Online Encyclopedia of Integer Sequences might give you the next terms, the name, references, generating functions,... . Since there is a steady progress in the enumeration of polyominoes we cite the corresponding sequences of [116] to give the reader the chance of getting the latest numbers.

Before we give the numbers we would like to encourage all mathematical authors to support this archive by contributing the integer sequences from their mathematical work. The author himself has submitted and extended over 600 sequences for this archive.

A.1 Number of square polyominoes

n	A0001055(n)	n	A0001055(n)	n	A0001055(n)
0	1	10	4.655	20	2.870.671.950
1	1	11	17.073	21	11.123.060.678
2	1	12	63.600	22	43.191.857.688
3	2	13	238.591	23	168.047.007.728
4	5	14	901.971	24	654.999.700.403
5	12	15	3.426.576	25	2.557.227.044.764
6	35	16	13.079.255	26	9.999.088.822.075
7	108	17	50.107.909	27	39.153.010.938.487
8	369	18	192.622.052	28	153.511.100.594.603
9	1.285	19	742.624.232		

Table A.1. A0001055 Polyominoes or square animals.

n	A001168(n)	n	A001168(n)
1	1	29	4.820.975.409.710.116
2	2	30	18.946.775.782.611.174
3	6	31	74.541.651.404.935.148
4	19	32	293.560.133.910.477.776
5	63	33	1.157.186.142.148.293.638
6	216	34	4.565.553.929.115.769.162
7	760	35	18.027.932.215.016.128.134
8	2.725	36	71.242.712.815.411.950.635
9	9.910	37	281.746.550.485.032.531.911
10	36.446	38	1.115.021.869.572.604.692.100
11	135.268	39	4.415.695.134.978.868.448.596
12	505.861	40	17.498.111.172.838.312.982.542
13	1.903.890	41	69.381.900.728.932.743.048.483
14	7.204.874	42	275.265.412.856.343.074.274.146
15	27.394.666	43	1.092.687.308.874.612.006.972.082
16	104.592.937	44	4.339.784.013.643.393.384.603.906
17	400.795.844	45	17.244.800.728.846.724.289.191.074
18	1.540.820.542	46	68.557.762.666.345.165.410.168.738
19	5.940.738.676	47	272.680.844.424.943.840.614.538.634
20	22.964.779.660	48	1.085.035.285.182.087.705.685.323.738
21	88.983.512.783	49	4.319.331.509.344.565.487.555.270.660
22	345.532.572.678	50	17.201.460.881.287.871.798.942.420.736
23	1.344.372.335.524	51	68.530.413.174.845.561.618.160.604.928
24	5.239.988.770.268	52	273.126.660.016.519.143.293.320.026.256
25	20.457.802.016.011	53	1.088.933.685.559.350.300.820.095.990.030
26	79.992.676.367.108	54	4.342.997.469.623.933.155.942.753.899.000
27	313.224.032.098.244	55	17.326.987.021.737.904.384.935.434.351.490
28	1.228.088.671.826.973	56	69.150.714.562.532.896.936.574.425.480.218

Table A.2. A001168 Fixed polyominoes with n cells.

A.2 Number of polyiamonds

n	A000577(n)	n	A000577(n)	n	A000577(n)
1	1	11	1.186	21	41.835.738
2	1	12	3.334	22	121.419.260
3	1	13	9.235	23	353.045.291
4	3	14	26.166	24	1.028.452.717
5	4	15	73.983	25	3.000.800.627
6	12	16	211.297	26	8.769.216.722
7	24	17	604.107	27	25.661.961.260
8	66	18	1.736.328	28	75.195.166.667
9	160	19	5.000.593		
10	448	20	14.448.984		

Table A.3. A000577 Triangular polyominoes (or polyiamonds) with n cells (turning over is allowed, holes are allowed, must be connected along edges).

n	AA001420(n)	n	A001420(n)	n	A001420(n)
1	2	11	14.016	21	501.994.070
2	3	12	39.169	22	1.456.891.547
3	6	13	110.194	23	4.236.446.214
4	14	14	311.751	24	12.341.035.217
5	36	15	886.160	25	36.009.329.450
6	94	16	2.529.260	26	105.229.462.401
7	250	17	7.244.862	27	307.942.754.342
8	675	18	20.818.498	28	902.338.712.971
9	1.838	19	59.994.514		
10	5.053	20	173.338.962		

Table A.4. A001420 Fixed 2-dimensional triangular-celled animals with n cells.

A.3 Number of polyhexes

n	A000228(n)	n	A000228(n)	n	A000228(n)
1	1	8	1.448	15	76.581.875
2	1	9	6.572	16	372.868.101
3	3	10	30.490	17	1.822.236.628
4	7	11	143.552	18	8.934.910.362
5	22	12	683.101	19	43.939.164.263
6	82	13	3.274.826	20	216.651.036.012
7	333	14	15.796.897		

Table A.5. A000228 Hexagonal polyominoes.

n	A001207(n)	n	A001207(n)
1	1	19	527.266.673.134
2	3	20	2.599.804.551.168
3	11	21	12.849.503.756.579
4	44	22	63.646.233.127.758
5	186	23	315.876.691.291.677
6	814	24	1.570.540.515.980.274
7	3.652	25	7.821.755.377.244.303
8	16.689	26	39.014.584.984.477.092
9	77.359	27	194.880.246.951.838.595
10	362.671	28	974.725.768.600.891.269
11	1.716.033	29	4.881.251.640.514.912.341
12	8.182.213	30	24.472.502.362.094.874.818
13	39.267.086	31	122.826.412.768.568.196.148
14	189.492.795	32	617.080.993.446.201.431.307
15	918.837.374	33	3.103.152.024.451.536.273.288
16	4.474.080.844	34	15.618.892.303.340.118.758.816
17	21.866.153.748	35	78.679.501.136.505.611.375.745
18	107.217.298.977		

Table A.6. A001207 Fixed hexagonal polyominoes with n cells.

A.4 Number of Benzenoids

In A.3 the numbers of polyhexes, polyominoes on the honeycomb lattice, were given. If we disallow these polyominoes to have holes we get another import (especially in chemistry) discrete structure - the *benzenoids* [30, 31, 75].

n	A018190(n)	n	A018190(n)
1	1	19	41.892.642.772
2	1	20	205.714.411.986
3	3	21	1.012.565.172.403
4	7	22	4.994.807.695.197
5	22	23	24.687.124.900.540
6	81	24	122.238.208.783.203
7	331	25	606.269.126.076.178
8	1.435	26	3.011.552.839.015.720
9	6.505	27	14.980.723.113.884.739
10	30.086	28	74.618.806.326.026.588
11	141.229	29	372.132.473.810.066.270
12	669.584	30	1.857.997.219.686.165.624
13	3.198.256	31	9.286.641.168.851.598.974
14	15.367.577	32	46.463.218.416.521.777.176
15	74.207.910	33	232.686.119.925.419.595.108
16	359.863.778	34	1.166.321.030.843.201.656.301
17	1.751.594.643	35	5.851.000.265.625.801.806.530
18	8.553.649.747		

Table A.7. A018190 Number of planar simply-connected polyhexes with n hexagons.

n		n	
1	1	19	502.709.028.125
2	3	20	2.468.566.918.644
3	11	21	12.150.769.362.815
4	44	22	59.937.663.454.017
5	186	23	296.245.438.278.258
6	813	24	1.466.858.366.128.911
7	3.640	25	7.275.229.222.292.218
8	16.590	26	36.138.633.393.334.038
9	76.663	27	179.768.675.964.165.939
10	358.195	28	895.425.672.624.735.867
11	1.688.784	29	4.465.589.678.921.947.602
12	8.022.273	30	22.295.966.620.155.816.954
13	38.351.973	31	111.439.693.993.112.940.196
14	184.353.219	32	557.558.620.919.353.655.115
15	890.371.070	33	2.792.233.438.943.251.452.902
16	4.318.095.442	34	13.995.852.369.729.891.369.431
17	21.018.564.402	35	70.212.003.186.716.473.817.832
18	102.642.526.470		

Table A.8. Fixed Benzenoids with n cells.

A.5 Number of 3-dimensional polyominoes

n	A000162(n)	n	A000162(n)	n	A000162(n)
1	1	6	166	11	2.522.522
2	1	7	1.023	12	18.598.427
3	2	8	6.922	13	138.462.649
4	8	9	48.311		
5	29	10	346.543		

Table A.9. A000162 3-dimensional polyominoes
(or polycubes) with n cells.

n	A001931(n)	n	A001931(n)	n	A001931(n)
1	1	7	23.502	13	3.322.769.321
2	3	8	162.913	14	24.946.773.111
3	15	9	1.152.870	15	188.625.900.446
4	86	10	8.294.738	16	1.435.074.454.755
5	534	11	60.494.549	17	10.977.812.452.428
6	3.481	12	446.205.905		

Table A.10. A001931 Fixed 3-dimensional polyominoes with n cells; lattice animals in the simple cubic lattice (6 nearest neighbors), face-connected cubes.

A.6 Number of polyominoes on Archimedean tessellations

The numbers given in this section are all taken from [14]. The eight Archimedean tessellations are depicted in Figure A.4 to A.11 where the cyclic sequences $(p_1, p_2, \ldots, p_q)$ represent the lists of the numbers of sides of all polygons surrounding any vertex in this order. In the tables n denotes the number of cells and $a(n)$ denotes the number of free polyominoes with n cells on the given tessellation.

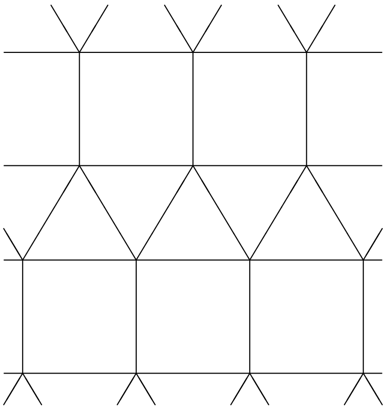

Figure A.1. (3,3,3,4,4).

n	a(n)	n	a(n)
1	2	9	2.822
2	3	10	9.207
3	5	11	30.117
4	13	12	99.708
5	32	13	331.219
6	96	14	1.106.870
7	281	15	3.710.728
8	891		

Table A.11. (3,3,3,4,4).

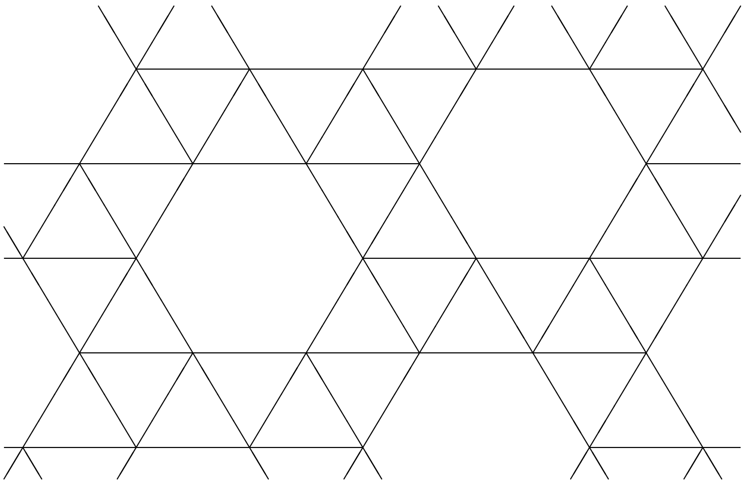

Figure A.2. (3,3,3,3,6).

n	a(n)	n	a(n)	n	a(n)	n	a(n)
1	3	5	69	9	8.943	13	1.345.840
2	3	6	228	10	31.164	14	4.758.782
3	7	7	762	11	108.840		
4	23	8	2.594	12	382.063		

Table A.12. (3,3,3,3,6).

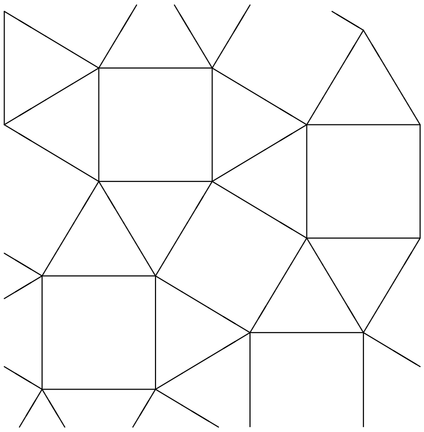

Figure A.3. (3,3,4,3,4).

n	a(n)	n	a(n)
1	2	9	2.161
2	2	10	6.690
3	4	11	20.881
4	10	12	65.593
5	28	13	207.171
6	79	14	657.301
7	233	15	2.093.785
8	705		

Table A.13. (3,3,4,3,4).

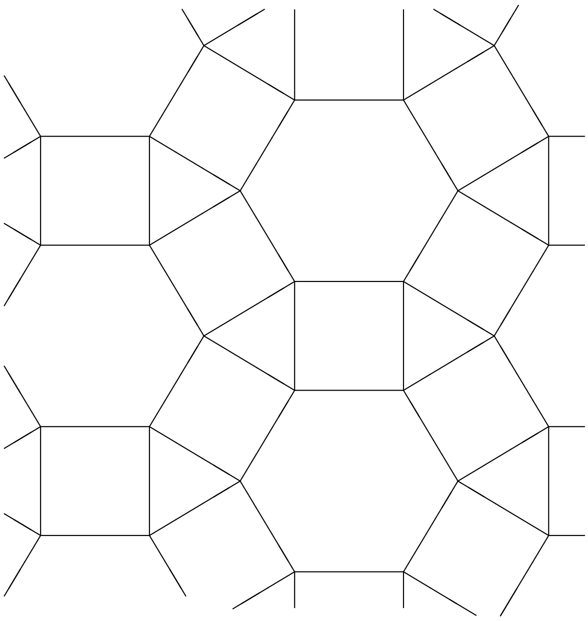

Figure A.4. (3,4,6,4).

n	a(n)
1	3
2	2
3	7
4	16
5	59
6	194
7	790
8	3.116
9	13.091
10	55.021
11	235.754
12	1.015.101
13	4.408.515

Table A.14. (3,4,6,4).

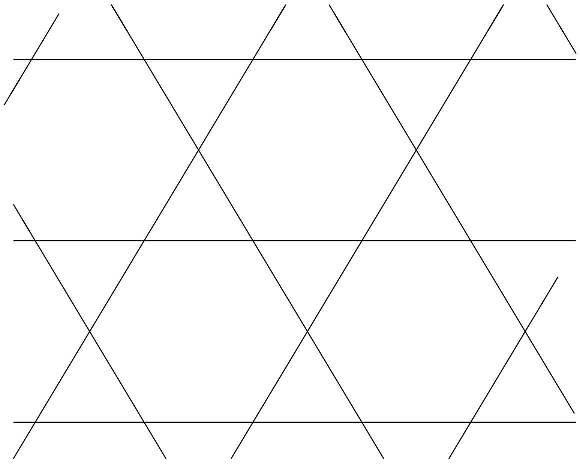

Figure A.5. (3,6,3,6).

n	a(n)
1	2
2	1
3	4
4	9
5	29
6	90
7	330
8	1.167
9	4.393
10	16.552
11	63.618
12	245.732
13	957.443
14	3.745.541

Table A.15. (3,6,3,6).

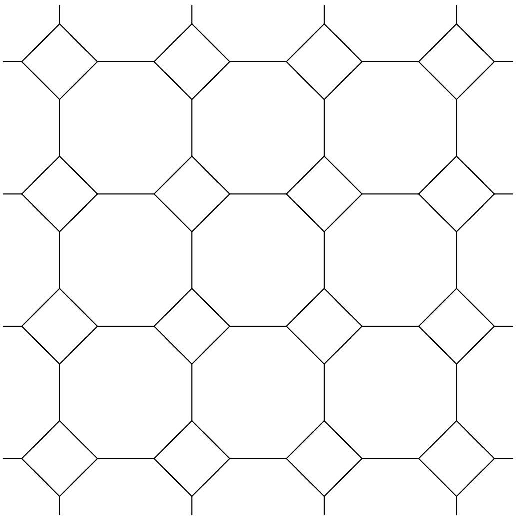

Figure A.6. (4,8,8).

n	a(n)	n	a(n)
1	2	7	1.914
2	2	8	9.645
3	7	9	50.447
4	21	10	266.992
5	90	11	1.432.165
6	388		

Table A.16. (4,8,8).

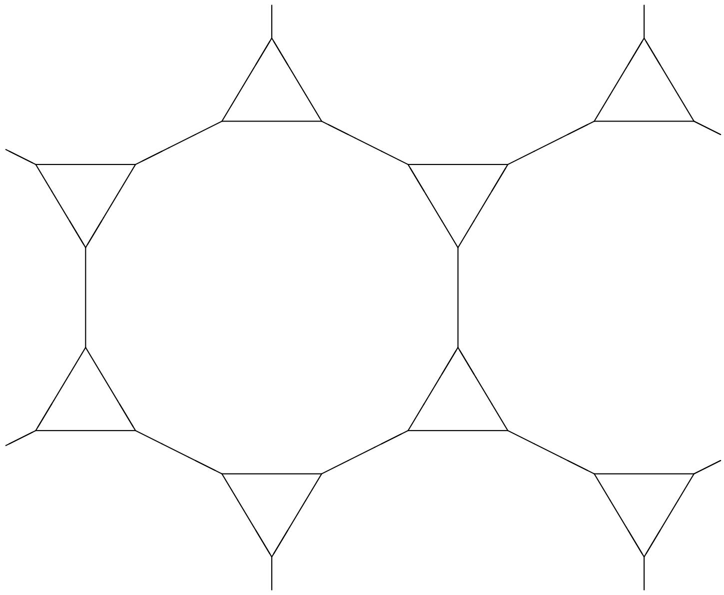

Figure A.7. (3,12,12).

n	a(n)	n	a(n)	n	a(n)	n	a(n)
1	2	4	35	7	5.949	10	1.541.542
2	2	5	173	8	37.198		
3	8	6	983	9	237.762		

Table A.17. (3,12,12).

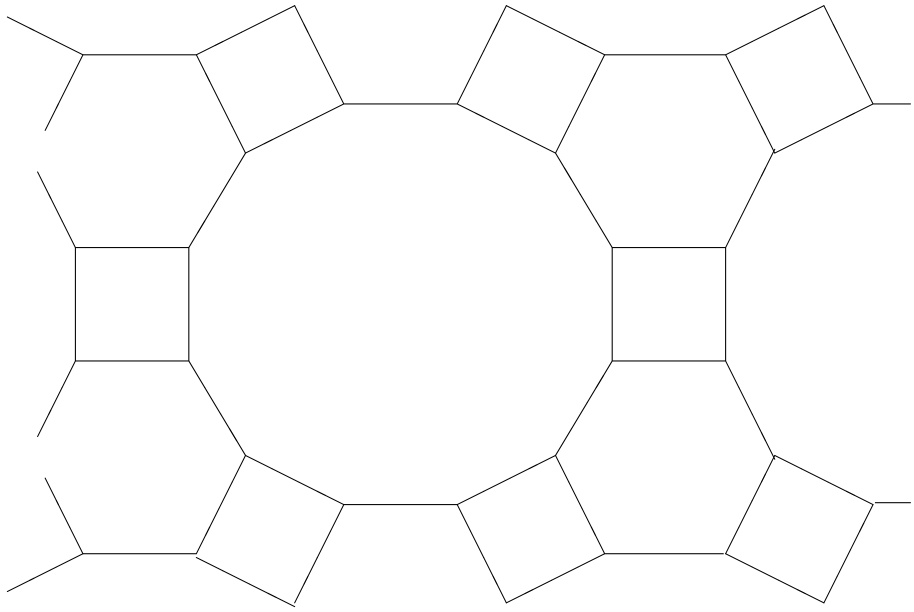

Figure A.8. (4,6,12).

n	a(n)	n	a(n)	n	a(n)	n	a(n)
1	3	4	49	7	7.796	10	1.697.278
2	3	5	255	8	45.876	11	10.472.378
3	14	6	1.327	9	278.002		

Table A.18. (4,6,12).

Appendix B

Deutsche Zusammenfassung

Ein **Polyomino** ist eine über Kanten verbundene Vereinigung von Zellen im ebenen Quadratgitter. Es gibt mindestens 3 unterschiedliche Möglichkeiten zu definieren, wann 2 Polyominoes als äquivalent betrachtet werden sollen. Man bezeichnet diese als fixe Polyominoes wenn äquivalente Polyominoes durch Verschiebungen auseinander hervorgehen, als chirale Polyominoes wenn äquivalente Polyominoes durch Verschiebungen oder Drehungen auseinander hervorgehen und man bezeichnet sie als freie Polyominoes wenn äquivalente Polyominoes durch Verschiebungen, Drehungen oder Spiegelungen auseinander hervorgehen. In der Literatur werden sie auch manchmal **animals** genannt, oder man spricht vom **Zellwachstums-Problem** [89, 112]. Für den Ursprung von Polyominoes zitiere ich in freier Übersetzung Klarner [90]: "Polyominoes haben eine lange Geschichte, die bis zum Anfang des 20. Jahrhunderts zurück geht. Aber einer breitern Öffentlichkeit bekannt gemacht wurden sie zunächst von Solomon Golomb [66-73] und Martin Gardner in seinen Kolumnen des *Scientific American*." Um die abstrakt definierten Polyominoes zu veranschaulichen sind in der Graphik 1.1 auf Seite 1 die Polyominoes aus höchstens fünf Quadraten dargestellt.

Es gibt mehrere Verallgemeinerungen von Polyominoes, z.B. Polyiamonds (kantenbenachbarte Vereinigungen von gleichseitigen Einheitsdreiecken) [13, 64, 78, 104, 120], Polyhexes (kantenbenachbarte Vereinigungen von regulären Einheitssechsecken) [11, 63, 64, 104], Polyabolos (kantenbenachbarte Vereinigungen von rechtwinkligen gleichschenkligen Einheitsdreiecken) [63], Polycubes (flächenbenachbarte Vereinigungen von Einheitswürfeln) [3, 105], usw. Des Weiteren kann man Polyominoes als Vereinigung von Zellen auf den Archimedischen Parkettierungen [14] definieren. In dieser Arbeit betrachten wir d-dimensionale Polyominoes als flächenbenachbarte Vereinigung von d-dimensionalen Einheitswürfeln. Falls nicht anders erwähnt, sind mit dem Terminus Polyominoes die freien Polyominoes gemeint.

Bevor die Sätze dieser Arbeit aufgelistet werden, sollen noch ein paar Anwendungen und Probleme von Polyominoes genannt werden. Der Ausdruck Zellwachstums-Problem suggeriert Anwendungen in Medizin und Biologie. Polyominoes sind nützlich für das *Ising Model* [32] mit dem man z.B. Nervennetzwerke, Vogelschwärme, schlagende Herzzellen, Atome, Proteinfaltungen, Membrane, soziales Verhalten, usw. modellieren kann. Weitere Anwendungen liegen auf dem Gebiet der Chemie und der Physik. Als Probleme mit Polyominoes seien das Abzählen von Polyominoes [1,2,4,5,10,15-31,34-38,40-62,65,74,76,79,82-89,92,94,96-115,119,121-124,126], das Erzeugen von Polyominoes [83, 85], Vollendungsspiele [11, 12, 13, 14, 78, 80] und extremale Polyominoes [8, 77, 81, 93] erwähnt. Im Anhang A werden exakte Anzahlen für einige Arten von Polyominoes aufgelistet.

Der Hauptteil dieser Arbeit handelt von Polyominoes mit maximalem Flächeninhalt der konvexen Hülle. In [8] wurde gezeigt, daß der maximale Flächeninhalt der konvexen Hülle eines Polyominoes aus n Einheitsquadraten $n + \frac{1}{2}\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$ beträgt. In dieser Arbeit wird die Vermutung aus [8] für den d-dimensionalen Fall bewiesen.

Satz 1. Das d-dimensionale Volumen der konvexen Hülle einer flächenbenachbarten Vereinigung von n Einheitshyperwürfeln ist höchstens

$$\sum_{I \subseteq \{1,\ldots,d\}} \frac{1}{|I|!} \prod_{i \in I} \left\lfloor \frac{n-2+i}{d} \right\rfloor .$$

Die Autoren von [8] fragten auch nach der Anzahl von verschiedenen Polyominoes aus n Quadraten mit maximalem Flächeninhalt der konvexen Hülle.

Satz 2. Die Anzahl $c_2(n)$ von Polyominoes im $\mathbb{R}^2$ maximalem Flächeninhalt der konvexen Hülle, bestehend aus n Einheitsquadraten, ist gegeben durch

$$n \equiv 0 \ (mod\ 4) : c_2(n) = \frac{n^3 - 2n^2 + 4n}{16},$$

$$n \equiv 1 \ (mod\ 4) : c_2(n) = \frac{n^3 - 2n^2 + 13n + 20}{32},$$

$$n \equiv 2 \ (mod\ 4) : c_2(n) = \frac{n^3 - 2n^2 + 4n + 8}{16},$$

$$n \equiv 3 \ (mod\ 4) : c_2(n) = \frac{n^3 - 2n^2 + 5n + 8}{32}.$$

Wenn man den maximalen Flächeninhalt der konvexen Hülle kennt, kann man fragen welche Flächeninhalte möglich sind. Für den zweidimensionalen Fall wird die Situation vollständig durch den nächsten Satz beschrieben.

Satz 3. Die Existenz eines zweidimensionalen Polyomino bestehend aus n Zellen mit einer Fläche a der konvexen Hülle ist äquivalent zu $a \in A_n$ mit

$$A_n = \begin{cases} \left\{ n + \frac{m}{2} \middle| m \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, m \in \mathbb{N}_0 - \{1\} \right\} & : \ falls\, n+1\, Primzahl, \\ \left\{ n + \frac{m}{2} \middle| m \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, m \in \mathbb{N}_0 \right\} & : \ sonst\,. \end{cases}$$

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