The inverse power index problem

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University of Bayreuth

EURO XXIV 12.07.2010
The European Economic Community

<table>
<thead>
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<th>votes</th>
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quota 70% — at least 12 out of 17 votes
The European Economic Community

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quota 70 % — at least 12 out of 17 votes

How powerful is Luxembourg?
Who cares about Luxembourg?

What is power?

Power is a measure of an entity’s ability to control the environment around itself, including the behavior of other entities.
Who cares about Luxembourg?

What is power?

Power is a measure of an entity’s ability to control the environment around itself, including the behavior of other entities.

a yes/no decision

(votes without Luxembourg / votes including Luxembourg)

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<tr>
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Who cares about Luxembourg?

What is power?
Power is a measure of an entity’s ability to control the environment around itself, including the behavior of other entities.

a yes/no decision

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Conclusion: Luxembourg has absolutely no power.
How to measure power

Shapley-Shubik power index (there are more power indices)

The power of a player is measured by the fraction of the possible voting sequences in which that player casts the deciding vote, that is, the vote that first guarantees passage or failure. The power index is normalized between 0 and 1.
How to measure power

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Example: $A \rightarrow 3$, $B \rightarrow 2$, $C \rightarrow 1$, $D \rightarrow 1$, quota= 4

\[
\begin{align*}
\text{ABCD} & \quad \text{ABDC} & \quad \text{ACBD} & \quad \text{ACDB} & \quad \text{ADBC} & \quad \text{ADCB} \\
\text{BACD} & \quad \text{BADC} & \quad \text{BCAD} & \quad \text{BCDA} & \quad \text{BDAC} & \quad \text{BDCA} \\
\text{CABD} & \quad \text{CABD} & \quad \text{CBAD} & \quad \text{CBDA} & \quad \text{CDAB} & \quad \text{CDBA} \\
\text{DABC} & \quad \text{DACB} & \quad \text{DBAC} & \quad \text{DBCA} & \quad \text{DCAB} & \quad \text{DCBA}
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\text{CABD} & \quad \text{CABD} & \quad \text{CBAD} & \quad \text{CBDA} & \quad \text{CDAB} & \quad \text{CDBA} \\
\text{DABC} & \quad \text{DACB} & \quad \text{DBAC} & \quad \text{DBCA} & \quad \text{DCAB} & \quad \text{DCBA}
\end{align*}
\]

\[
\begin{align*}
\text{Pow}(A) = \frac{12}{24}, \quad \text{Pow}(B) = \frac{4}{24}, \quad \text{Pow}(C) = \frac{4}{24}, \quad \text{Pow}(D) = \frac{4}{24}
\end{align*}
\]
Shapley-Shubik power index

**Definition**
A voter $i$ is a *swing* for coalition $S$ if $S \cup \{i\}$ is winning but $S$ is not.

**Example:** $A \rightarrow 3$, $B \rightarrow 2$, $C \rightarrow 1$, $D \rightarrow 1$, quota = 4
Coalition $S = \{A\}$ is a swing for voter $B$ since $\{A, B\}$ is winning but $\{A\}$ is losing.
Shapley-Shubik power index

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A voter $i$ is a **swing** for coalition $S$ if $S \cup \{i\}$ is winning but $S$ is not.

Example: $A \rightarrow 3$, $B \rightarrow 2$, $C \rightarrow 1$, $D \rightarrow 1$, quota= 4
Coalition $S = \{A\}$ is a swing for voter $B$ since $\{A, B\}$ is winning but $\{A\}$ is losing.

Lemma

$$Pow(i) = \frac{1}{n!} \cdot \sum_{S \subseteq N \setminus \{i\} \text{ is swing for } i} |S|! \cdot (n - |S| - 1)!$$
Weighted voting games

Definition
An $n$-player weighted voting game is represented by a list $[q; w_1, w_2, \ldots, w_n]$ of non-negative real numbers: $[4; 3, 2, 1, 1]$

A coalition $C \subseteq \{1, \ldots, n\}$ is winning if $\sum_{i \in C} w_i \geq q$ and losing otherwise.
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Alternatives for The European Economic Community

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Which voting game is more appropriate?
Penrose square-root law

How many votes for Poland? – One man one vote?
Penrose square-root law

One man one vote?

In a fair assignment of voting weights the voting power of a state should be proportional to the square-root of the population of this state.

Lionel Penrose, 1946
Penrose square-root law

One man one vote?

In a **fair** assignment of voting weights the voting power of a state should be proportional to the square-root of the population of this state.

Lionel Penrose, 1946

### Treaty of Nice (2001)

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Optimization problem – the key motivation

For \( n \) players, a vector \( I \) of the ideal power distribution, and a norm \( \| \cdot \| \) determine a weighted voting game \( W \) such that the difference

\[ \| I - \text{Pow}(W) \| \]

between the ideal power distribution and the realized power distribution is minimized.
Optimization problem – the key motivation

For \( n \) players, a vector \( I \) of the *ideal* power distribution, and a norm \( \| \cdot \| \) determine a weighted voting game \( W \) such that the difference

\[
\| I - \text{Pow}(W) \|
\]

between the ideal power distribution and the realized power distribution is minimized.

As things are now

Only heuristics are used: fixed point iteration

\[
W = W + \lambda \cdot (I - \text{Pow}(W))
\]

on the voting weights Dennis Leech, 2002.
Obtaining a finite problem

Problem
The weights and the quota of a weighted voting game are not unique: \([4; 3, 2, 1, 1]\), \([11; 9, 5, 5, 4]\), and \([q; q - 2, x, x, x]\) for \(q \geq 6\) and \(\left\lceil \frac{q}{3} \right\rceil \leq x \leq \left\lfloor \frac{q - 1}{2} \right\rfloor\) represent the same game.
Obtaining a finite problem

Problem
The weights and the quota of a weighted voting game are not unique: [4; 3, 2, 1, 1], [11; 9, 5, 5, 4], and [q; q − 2, x, x, x] for q ≥ 6 and \( \left\lfloor \frac{q}{3} \right\rfloor \leq x \leq \left\lceil \frac{q-1}{2} \right\rceil \) represent the same game.

Underlying discrete structure
The (subset-) minimal winning coalitions of [4; 3, 2, 1, 1] are given by \{A, B\}, \{A, C\}, \{A, D\}, \{B, C, D\}. (The maximal losing coalitions are given by \{A\}, \{B, C\}, \{B, D\}, \{C, D\}. So each weighted voting game is a monotone Boolean (threshold) function \( \Upsilon : \{0, 1\}^n \to \{0, 1\} \) (a.k.a. simple game if surjective).

(Preliminary) Optimization algorithm
Loop over the finite list of weighted voting games an pick the one which maximizes the given target function.
An ILP formulation

Setting

- desired power distribution $\sigma = (\sigma_1, \ldots, \sigma_n)$,
- set $N = \{1, \ldots, n\}$ of voters, $w_A = (|A|! \cdot (n - |A| - 1))$,
- minimize the $\|Pow(\chi) - \chi\|_1$ for simple games $\chi$
An ILP formulation

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- set $N = \{1, \ldots, n\}$ of voters, $w_A = (|A|! \cdot (n - |A| - 1))$,
- minimize the $\|\text{Pow}(\chi) - \chi\|_1$ for simple games $\chi$

Variables

- $x_A \in \{0, 1\}$: is coalition $A \subseteq N$ winning?
- $y_A \in \{0, 1\}$: is coalition $A$ a swing for voter $i$?
- $p_i$: Shapley-Shubik power of voter $i$
- $d_i$: $|p_i - \sigma_i|$
An ILP formulation (cont.)

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{n} d_i \\
\text{s.t.} & \quad \sigma_i - d_i \leq p_i \leq \sigma_i + d_i \quad \forall i \in N, \\
& \quad p_i = \sum_{A \in N \setminus \{i\}} w_A \cdot y_{i,A} \quad \forall i \in N, \\
& \quad y_{i,A} = x_{A \cup \{i\}} - x_A \quad \forall i \in N, A \subseteq N \setminus \{i\}, \\
& \quad x_A \geq x_{A \setminus \{j\}} \quad \forall \emptyset \neq A \subseteq N, j \in A, \\
& \quad x_\emptyset = 0 \\
& \quad x_N = 1 \\
& \quad x_A \in \{0, 1\} \quad \forall A \subseteq N, \\
& \quad y_{i,A} \in \{0, 1\} \quad \forall i \in N, A \subseteq N \setminus \{i\}, \\
& \quad d_i, p_i \geq 0 \quad \forall i \in N.
\end{align*}
\]
Power vectors which are hard to approximate

Problem
Find a simple game $\chi$ such that $\|\text{Pow}(\chi) - \sigma_n\|_1$ is minimized for $\sigma_n = (0.75, 0.25, 0, \ldots, 0)$. $n$ voters
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For $2 \leq n \leq 14$ voters we have

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for all simple games (weighted voting games).
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Conjecture
The above inequality is true for all $n \geq 2$ and for each power distribution $\sigma$ there exists a weighted voting games $\chi$ with

$$\|\text{Pow}(\chi) - \sigma\|_1 \leq \frac{1}{3}.$$
The result of Alon and Edelmann (2009)

**Theorem**

Let $n > k$ be positive integers, let $\varepsilon < \frac{1}{k+1}$ be a positive real, and let $\chi$ be a simple game for $n$ voters. If

$$\sum_{i=k+1}^{n} B(\chi, i) \leq \varepsilon,$$

then there exists a simple game $\chi'$ for $k$ voters such that

$$\||B(\chi) - B(\chi')||_1 =$$

$$\sum_{i=1}^{k} |B(\chi, i) - B(\chi', i)| + \sum_{i=k+1}^{n} |B(\chi, i) - 0| \leq \frac{(2k + 1)\varepsilon}{1 - (k + 1)\varepsilon} + \varepsilon.$$
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**Application**
Using $k = 2$ and $\varepsilon = \frac{1}{18}$ one can deduce  
$$\|B(\chi) - \sigma_n\|_1 \geq \frac{1}{9}.$$
Observation
Consider the weights $w_1 \geq w_2 \geq \cdots \geq w_n$ and let $\tau = (i, j)$ be a transposition with $i \leq j$. If a coalition $\tilde{c}$ contains voter $j$, then $\Upsilon(\tau(\tilde{c})) \geq \Upsilon(\tilde{c})$ and we write $\tau(\tilde{c}) \succeq \tilde{c}$. (We have shifted voter $j$ to voter $i$.) Simple games satisfying this partial order are called complete games.
Shift order on coalitions

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Consider the weights $w_1 \geq w_2 \geq \cdots \geq w_n$ and let $\tau = (i, j)$ be a transposition with $i \leq j$. If a coalition $\tilde{c}$ contains voter $j$, then $\Upsilon(\tau(\tilde{c})) \geq \Upsilon(\tilde{c})$ and we write $\tau(\tilde{c}) \succeq \tilde{c}$. (We have shifted voter $j$ to voter $i$.) Simple games satisfying this partial order $\succeq$ are called complete games.

Remark
The class of complete simple games is a super class of the weighted voting games. One can decide whether a complete simple game is a weighted voting game by solving a linear program. (The presented ILP can be adjusted by adding the corresponding inequalities.)
Enumeration results

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<td>0.5 s</td>
<td>2.25 m</td>
<td>4.3 d</td>
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Table: Number $cs(n)$ of complete simple games and $wm(n)$ of weighted voting games. (new results)

Remark
We have used a reimplementation of the dual simplex algorithm (tableaux representation) to solve the intermediate linear programs. In average less than 9 micro seconds per linear program where needed. (In this time the used machine can perform only 2500 integer additions in a loop.)
7 voters: Power distribution for voters 1 and 2
(projection of the possible power vectors)

Banzhaf power index

Shapley-Shubik power index

\[ i < j: \quad p_i \geq p_j \quad \text{and} \quad ip_i + (j - i)p_j \leq 1 \]

\[ i = 1 < j: \quad (j - 1)p_1 + (n + 1 - j)p_j \geq 1 \]
8 voters: Power distribution for voters 1 and 2

Banzhaf power index

Shapley-Shubik power index
9 voters: Power distribution for voters 1 and 2

Banzhaf power index

Shapley-Shubik power index
Future research

- prove the $\frac{1}{3}$-approximation conjecture
- combine the ILP- and the exhaustive enumeration approach
- try to characterize the empty regions, where power distributions are hard to approximate

Thank you very much for your attention!
Dedekind’s problem

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</table>

**Table**: Number of monotone Boolean functions.
Shift order on coalitions

Observation
Consider the weights \( w_1 \geq w_2 \geq \cdots \geq w_n \) and let \( \tau = (i, j) \) be a transposition with \( i \leq j \). If a coalition \( \tilde{c} \) contains voter \( j \), then \( \Upsilon(\tau(\tilde{c})) \geq \Upsilon(\tilde{c}) \) and we write \( \tau(\tilde{c}) \succeq \tilde{c} \). (We have shifted voter \( j \) to voter \( i \).) \( \succeq \) is a partial order.
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Definition
If neither \( \tilde{c}_1 \preceq \tilde{c}_2 \) nor \( \tilde{c}_1 \succeq \tilde{c}_2 \) is valid then we write \( \tilde{c}_1 \bowtie \tilde{c}_2 \). (We say that these coalitions are incomparable.)

Example
\[
\begin{align*}
[1100] & \succeq [1010] \succeq [1001] & \text{and} & & [1100] & \bowtie [0111].
\end{align*}
\]
The Hasse diagram induced by the shift order

Example
Complete simple games

Definition
A complete simple game is a set $\emptyset \neq \mathcal{W} \subseteq \{0, 1\}^n \setminus 0$ where $\tilde{w}_1 \succ \tilde{w}_2$ for all $\tilde{w}_1 \neq \tilde{w}_2 \in \mathcal{W}$.

A coalition $\tilde{c} \in \{0, 1\}^n$ is winning if and only if there exists an $\tilde{w} \in \mathcal{W}$ with $\tilde{c} \preceq \tilde{w}$.
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Example
The shift-minimal winning coalitions of $[4; 3, 2, 1, 1]$ ($\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C, D\}$) are given by $\{[1001], [0111]\}$.

(The shift-maximal losing coalitions are given by $\{[1000], [0110]\}$.)

Remark
The class of complete simple games is a super class of the weighted voting games.
Weighted voting games

Lemma

A complete simple game is a weighted voting game if and only if

$$\tilde{w}_i \cdot x > \tilde{l}_j \cdot x \quad \text{or} \quad \tilde{w}_i \cdot x \geq y$$

$$\tilde{l}_j \cdot x < y$$

$$x_1 \geq \cdots \geq x_n \geq 0 \quad x_1 \geq \cdots \geq x_n \geq 0, \ y > 0$$

has a solution for all shift-minimal winning coalitions $\tilde{w}_i$ and all shift-maximal losing coalitions $\tilde{l}_j$. ⇔ linear program
Exploiting incomplete information

Partial determination of losing coalitions
Suppose we are given a subset $\mathcal{W}'$ of the shift-minimal winning coalitions $\mathcal{W}$. Can we determine a set $\mathcal{L}'$ such that for each $l' \in \mathcal{L}'$ there is a shift-maximal losing coalition $l \in \mathcal{L}$ such that $l' \preceq l$?
Partial determination of losing coalitions

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If the linear program for the voting weights with $\mathcal{W}'$, $\mathcal{L}'$ instead of $\mathcal{W}$, $\mathcal{L}$ is infeasible, then $\mathcal{W}'$ can not be a subset of the shift-minimal winning coalitions of a weighted voting game.
Exploiting incomplete information (cont.)

Lemma, K. 2009
The set

\[ \mathcal{L}' := \left\{ w - e_i \mid w \in \mathcal{W}', w - e_i \in \{0, 1\}^n \right\} \cup \right. \\
\left. \left\{ w - e_i + e_{i+1} \mid w \in \mathcal{W}', w - e_i + e_{i+1} \in \{0, 1\}^n \right\} \right. \\

is such a partial losing set, where \( e_i \) denotes the \( i \)th unit-vector.

Example
For \( \mathcal{W}' = \{[1001], [0111]\} \) we obtain

\[ \mathcal{L}' = \{[1000], [0110], [0101], [0011], [0001]\}, \]

which can be condensed to \( \mathcal{L}'' = \{[1000], [0110]\} \).
Enumeration of weighted voting games (cont.)

A more sophisticated enumeration algorithm, K. 2009

- generate sets of shift-minimal winning coalitions $\mathcal{W}'$ of complete simple games using orderly generation
- in each node
  - determine a condensed but large partial losing set $\mathcal{L}'$
  - check, whether the linear program corresponding to $\mathcal{W}', \mathcal{L}'$ has a real-valued solution for the voting weights, using an highly optimized implementation of the standard dual simplex algorithms based on tableau’s reusing previous tableau’s
  - if no such solution exists prune the search-tree
  - otherwise determine the set $\mathcal{L}$ of shift-maximal losing vectors for $\mathcal{W}'$, solve the corresponding linear program, and eventually output a weighted voting game $(\mathcal{W}', \mathcal{L})$. 