Limit results for power indices

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Agenda

Introduction

Limit results

Approximating power by weights

Open problems
## European Economic Community – Treaty of Rome, 1957

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Quota 70% — at least 12 out of 17 votes

How powerful or influential are Luxembourg and Belgium?
The influence of Luxembourg

A Yes/No-decision

(Votes without Luxembourg / Votes including Luxembourg)

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Conclusion:

- Luxembourg has no power or influence
- Power is in general not proportional to weights
How to measure power?

Shapley-Shubik power index (there are others, e.g., Banzhaf, Public Good, Nucleolus)

Defined as the fraction of the possible voting sequences in which a player casts the deciding vote.

Example: $A \rightarrow 3$, $B \rightarrow 2$, $C \rightarrow 1$, $D \rightarrow 1$, Quota = 4

\[
\begin{align*}
  SSI(A) &= \frac{12}{24},
  SSI(B) &= \frac{4}{24},
  SSI(C) &= \frac{4}{24},
  SSI(D) &= \frac{4}{24}
\end{align*}
\]
Weighted voting games

**Definition**

An \( n \)-player weighted voting game \([q; w_1, w_2, \ldots, w_n]\) consists of a quota \( q \in \mathbb{R}_{>0} \) and voting weights \( w_i \in \mathbb{R}_{\geq 0} \): \([4; 3, 2, 1, 1]\).

A coalition \( T \subseteq \{1, \ldots, n\} \) wins if \( \sum_{i \in T} w_i \geq q \) and loses otherwise.
Alternative rules for the European Economic Community

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Which rule is the most appropriate?
The inverse problem – Computing the *fair* voting weights

**Optimization problem**

For *n* players, a vector *I* representing the *ideal* power distribution, a *power index* $Pow(\cdot)$, and a *norm* $\| \cdot \|$, determine a weighted voting game $W$, such that the difference

$$\| I - Pow(W) \|$$

between the desired and the obtained power distribution is *minimized*.

~~ My personal main motivation for doing research in cooperative game theory.
## Penrose’s limit theorem


Given a weighted voting game \( \left[ \sum_{i=1}^{n} w_i; w_1, \ldots, w_n \right] \) satisfying certain technical conditions, then

\[
B(i) \approx w_i \cdot \sqrt{\frac{2}{\pi \sum_{i=1}^{n} w_i^2}}.
\]

For increasing number of voters

\[
\frac{B(i)}{B(j)} \to \frac{w_i}{w_j}.
\]

**Conclusion:** In the *limit*, power is proportional to weights.
Rigorous results for the Shapley-Shubik index

Theorem – Abraham Neyman, Renewal theory for sampling without replacement, Annals of Probability 10(2), 1982

Let \( n \in \mathbb{N} \), \( N = \{ 1, \ldots, n \} \), \( 0 < q < 1 \), \( w \in \mathbb{R}_{\geq 0}^n \) with \( \| w \|_1 = 1 \), and \( P(i, q) \) be the probability that in a random order of \( N \), \( i \) is the first element in the order for which the \( w \)-accumulated sum exceeds \( q \). For every \( \varepsilon > 0 \) there exist constants \( \delta > 0 \) and \( K > 0 \) such that if \( \rho = \max_{i \in N} w_i < \delta \), and \( K \rho < q < 1 - K \rho \) then \( \| P(\cdot, q) - w \|_1 < \varepsilon \).
A limit result for the Nucleolus
(→ Maria Montero)

Theorem – K., Stefan Napel, and Andreas Nohn, The Nucleolus of Large Majority Games, Economics Letters 123, 2014

Let $x$ be an element of the nucleus (superset of the nucleolus) of a weighted voting game $[q; w]$ with $0 < q < 1$, $w_1 \geq \cdots \geq w_n \geq 0$, and $\sum_{i=1}^{n} w_i = 1$. Then

$$\|x - w\|_1 \leq \frac{2w_1}{\min(q, 1-q)}.$$

Corollary

Let $[q^1; w^1], [q^2; w^2], \ldots$ be a sequence of weighted voting games with $\|w^i\|_1 = 1$, $\varepsilon < q^i < 1 - \varepsilon$ for a fix $\varepsilon > 0$, and $\lim_{i \to \infty} \|w^i\|_\infty = 0$, then the nucleolus of $[q^i; w^i]$ tends to $w^i$, with respect to the $\| \cdot \|_1$-norm, as $n$ approaches infinity.
What we really need

\[ \| p - w \|_1 = \sum_{i=1}^{n} |p_i - w_i| \leq \frac{c\Delta}{\min\{q, 1 - q\}}, \]

for weighted game \([q; w]\) with \(\| w \|_1 = 1\), power \(p = \varphi([q; w])\), maximum relative weight \(\Delta\), and constant \(c = c(\varphi)\).

A bound for the approximation error in terms of easy invariants.

Note that \(\| x \|_{\infty} \leq \| x \|_2 \leq \| x \|_1\).

Applications

- approximation of power in large voting games (larger than the IMF) with explicit error bounds
- statements about power distributions in situations with incomplete or uncertain information
- limit results are just corollaries
General results for all power indices?

Lemma

Let \( n \in \mathbb{N}_{>0}, \, q, \overline{q} \in (0, 1], \, w, \overline{w} \in \mathbb{R}_{\geq 0}^n \) with \( \|w\|_1 = \|\overline{w}\|_1 = 1 \) and \([q; w] = [\overline{q}; \overline{w}], \| \cdot \| \) be an arbitrary norm on \( \mathbb{R}^n \) and \( \varphi \) be a power index, i.e., a mapping from the set of weighted games (on \( n \) voters) into \( \mathbb{R}_{\geq 0}^n \) (with possible additional structure) then we have

\[
\max \{ \|w - \varphi ([q; w])\|, \|\overline{w} - \varphi ([\overline{q}; \overline{w}]\| \} \geq \frac{\|w - \overline{w}\|}{2}.
\]

Proof

Using the triangle inequality yields

\[
\|w - \varphi ([q; w])\| + \|\overline{w} - \varphi ([\overline{q}; \overline{w}]\| \geq \|w - \overline{w}\| \text{ from which we can conclude the stated inequality.}
\]
Proposition

Let $\varphi$ be a power index, i.e., a mapping from the set of weighted games (on $n$ voters) into $\mathbb{R}^n_{\geq 0}$.

(i) For each $q \in (0, 1]$ and each integer $n \geq 2$ there exists a weighted game $\nu$ with $n$ voters that permits a representation $[q; w] = \nu$, where $w \in \mathbb{R}^n_{\geq 0}$ and $\|w\|_1 = 1$, such that 
$$\|w - \varphi([q; w])\|_\infty \geq \frac{1}{6} \quad \text{and} \quad \|w - \varphi([q; w])\|_1 \geq \frac{1}{3}.$$ 

(ii) For each $\Delta \in (0, 1)$ and each integer $n \geq \frac{4}{3\Delta} + 6$ there exists a weighted game $\nu$ with $n$ voters that permits a representation $[q; w] = \nu$, where $q \in (0, 1]$, $w \in \mathbb{R}^n_{\geq 0}$, 
$$\|w\|_1 = 1, \quad \text{and} \quad \Delta(w) = \Delta,$$ 
such that 
$$\|w - \varphi([q; w])\|_1 \geq \frac{1}{3}, \quad \text{and} \quad \|w - \varphi([q; w])\|_\infty \geq \Delta/4.$$
The best we can hope for

**Theorem**

Let \( \varphi \) be a power index, i.e., a mapping from the set of weighted games (on \( n \) voters) into \( \mathbb{R}^n_{\geq 0} \). For each \( 0 < \hat{q} < 1 \) there exists a weighted game \([q; w]\) which permits a normalized representation with quota \( \hat{q} \) such that

\[
\| w - \varphi([q; w]) \|_1 \geq \frac{1}{10} \cdot \min \left\{ 2, \frac{\Delta(w)}{\min\{\hat{q}, 1 - \hat{q}\}} \right\}.
\]
An approximation result

**Theorem**

Let \( w \in \mathbb{R}^n_{\geq 0} \) with \( \| w \|_1 = 1 \) and \( 0 < q < 1 \). If a power index \( \varphi \) permits the existence of a quota \( q' \in (0, 1) \) such that \([q'; \varphi([q; w])] = [q; w]\), i.e., that the power vector of the given weighted game can be completed to a representation of the same game, then

\[
\| w - \varphi([q; w]) \|_1 \leq \frac{4\Delta(w)}{\min\{q, 1 - q\}}.
\]

**Remark**

Applies to the **minimum sum representation index** or one of the power indices based on **averaged representations** (→ Serguei Kaniovski) for all weighted games and for the **Penrose-Banzhaf index** for all **spherically separable games** (→ Bill Zwicker). Also applicable for a simple market value bargaining model (→ Bernie Grofman).
Numerical investigations

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<th>Penrose-Banzhaf</th>
<th>Johnston</th>
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Table: Necessary constant $c$ for the approximation of the normalized minimum sum integer representation.
The case of the Penrose-Banzhaf index

Parameterized example: \( w_i = 2 \) for \( 1 \leq i \leq m \), \( w_i = 1 \) for \( m + 1 \leq i \leq 2m \), \( q = \alpha \cdot 3m; m \to \infty \)

Figure: Deviation between weights and the Penrose-Banzhaf index.

\[ f(x) = \left| x - \frac{1}{2} \right|^3 \cdot \frac{8}{3} \]
Open problems

\[ \|p - w\|_1 = \sum_{i=1}^{n} |p_i - w_i| \leq \frac{c\Delta}{\min\{q, 1-q\}} \]  (1)

- Prove that the Shapley-Shubik index satisfies Inequality (1)
- Prove that the Penrose-Banzhaf index satisfies Inequality (1) if \( q = \frac{1}{2} \)
- Determine classes of weighted games larger than unanimity games that another power index satisfies Inequality (1)
- Study the \textit{oceanic} case
Thanks

Thank you for the invitation and your attention!
### The nucleolus

**Definition**

An **imputation** for simple game \((N, \nu)\), where \(N = \{1, \ldots, n\}\), is a vector \(x = (x_i)_{i \in N}\) with \(x_i \geq 0\) for all \(i \in N\) and \(x(N) \leq 1\).

**Example**

For \([4; 3, 2, 1, 1]\) the vector \((\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})\) is an imputation.

**Definition**

For any coalition \(S\) and any imputation \(x\), the **excess** of \(S\) at \(x\) is 
\[ e(S, x) = \nu(S) - x(S). \]

**Example**

\[
\begin{align*}
e(\{1, 4\}, x) &= 1 - \frac{1}{2} = \frac{1}{2} \\
e(\{3, 4\}, x) &= 0 - \frac{1}{3} = -\frac{1}{3}
\end{align*}
\]
The nucleolus (cont.)

Definition
For any imputation $x$, let $S_1, \ldots, S_{2^n}$ be an ordering of all coalitions such that the excess at $x$ is weakly decreasing. The excess vector is the vector $E(x) = (e(x, S_k))_{1 \leq k \leq 2^n}$ with (weakly) decreasing components.

Example
For $v = [4; 3, 2, 1, 1]$ and $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$ the excess vector is given by

- $\{1, 3\}, \{1, 4\} \mapsto \frac{1}{2}$;
- $\{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\} \mapsto \frac{1}{3}$;
- $\{1, 2, 3\}, \{1, 2, 4\} \mapsto \frac{1}{6}$;
- $\emptyset, \{1, 2, 3, 4\} \mapsto 0$;
- $\{3\}, \{4\} \mapsto -\frac{1}{6}$;
- $\{1\}, \{2\}, \{3, 4\} \mapsto -\frac{1}{3}$;
- $\{2, 3\}, \{2, 4\} \mapsto -\frac{1}{2}$.
The nucleolus (cont.)

**Definition**

Imputation $x$ is **lexicographically less** than imputation $y$ if $E_k(x) < E_k(y)$ for the smallest component $k$ with $E_k(x) \neq E_k(y)$.

**Example**

$x := \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) \prec y := \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$ since $E(x) = \left(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \ldots\right)$ and $E(y) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\right)$.

**Definition**

The **nucleolus** is then uniquely defined as the lexicographically minimal imputation. $\sim \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$ is the nucleolus for $[4; 3, 2, 1, 1]$.

**Remark:** $[4; 3, 2, 1, 1] = [3; 2, 1, 1, 1]$, i.e. the relative weights equal the nucleolus.
The oceanic case

Example
Consider a stock corporation whose shares are owned by three major stockholders owning 35%, 34%, and 17%, respectively. The remaining 14% are widely spread. Assuming that decisions are made by a simple majority rule.

Resulting power distribution for the major stockholders
Any two major stockholders can adopt a proposal, while the private shareholders together with an arbitrary major stockholder need further affirmation. Very special situation.
Auxiliary problem for the nucleolus

Set $N$ of voters, small ones collected in $O$, and a few large ones collected in $N \setminus O$. Let $\bar{w}$ be the vector of relative weights, $\bar{q}$ be the relative quota, $\alpha = \bar{w}(O)$ the weight mass of the small voters, and $x$ be an optimal solution of

\[
\begin{align*}
\min y \\
y + \sum_{i \in S} x_i &\geq 1 \\
y + \frac{\bar{q} - \bar{w}(S)}{\alpha} \cdot \beta + \sum_{i \in S} x_i &\geq 1 \\
\beta + \sum_{i \in N \setminus O} x_i & = 1 \\
x_i &\in \mathbb{R}_{\geq 0} \\
\beta &\in \mathbb{R}_{\geq 0}
\end{align*}
\]

subject to

\[
\begin{align*}
\forall S \subseteq N \setminus O : \bar{w}(S) &\geq \bar{q} \\
\forall S \subseteq N \setminus O : \bar{q} - \alpha &\leq \bar{w}(S) < \bar{q}
\end{align*}
\]

We claim that $x_i^* = x_i$ for $i \in N \setminus O$ and $x_i^* = \bar{w}_i \cdot \frac{\beta}{\alpha}$ is a good approximation for the nucleolus $\bar{x}$ of $[\bar{q}; \bar{w}]$. More precisely, we conjecture that there exists a constant $c \in \mathbb{R}_{> 0}$ such that

\[
\|\bar{x} - x^*\|_1 \leq \frac{c \Delta_O}{\min \{|\bar{q} - \bar{w}(S)| : S \subseteq N \setminus O\}},
\]

where $\Delta_O = \max \{\bar{w}_i : i \in O\}$ is the maximum relative weight of a small voter.
Lemma

Let \( w \in \mathbb{R}^n_{\geq 0} \) with \( \|w\|_1 = 1 \), \( 0 < q < 1 \) and \( \varphi \) be a symmetric, efficient, and positive power index. If \( \|\varphi([q; w]) - w\|_1 \leq \varepsilon \), then

\[
1 - \frac{\varepsilon}{2\alpha} \leq \frac{\varphi_i([q; w])}{w_i} \leq 1 + \frac{\varepsilon}{2\alpha}
\]

for all \( 1 \leq i \leq n \) with \( w_i > 0 \), where \( \alpha = w(S) > 0 \) and \( S := \{1 \leq j \leq n : w_i = w_j\} \).
Lemma

If \( w_i, w_j, \varphi_i, \varphi_j \in \mathbb{R}_{>0}, \varepsilon_i, \varepsilon_j \in [0, 1) \) with \( 1 - \varepsilon_i \leq \frac{\varphi_i}{w_i} \leq 1 + \varepsilon_i \) and \( 1 - \varepsilon_j \leq \frac{\varphi_j}{w_j} \leq 1 + \varepsilon_j \), then

\[
\frac{1 - \varepsilon_i}{1 + \varepsilon_j} \leq \frac{w_i}{w_j} \cdot \frac{\varphi_j}{\varphi_i} \leq \frac{1 + \varepsilon_i}{1 - \varepsilon_j}
\]

and

\[
\left| \frac{\varphi_i}{w_i} - \frac{\varphi_j}{w_j} \right| \leq \varepsilon_i + \varepsilon_j.
\]
The Laakso-Taagepera index

**Definition**
For \( w \in \mathbb{R}^n_\geq 0 \) with \( w \neq 0 \) the Laakso-Taagepera index is given by

\[
L(w) = \left( \frac{\sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} w_i^2} \right)^2.
\]

**Lemma**
For \( w \in \mathbb{R}^n_\geq 0 \) with \( \|w\|_1 = 1 \), we have

\[
\frac{1}{\Delta} \leq \frac{1}{\Delta (1 - \alpha(1 - \alpha)\Delta)} \leq L(w) \leq \Delta^2 + \frac{(1-\Delta)^2}{n-1} \leq \frac{1}{\Delta^2}
\]
for \( n \geq 2 \), where \( \alpha := \frac{1}{\Delta} - \left\lfloor \frac{1}{\Delta} \right\rfloor \in [0, 1) \). If \( n = 1 \), then \( \Delta = L(w) = 1 \).
References


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