Integral point sets in Euclidean spaces

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Integral point sets

**Definition**

An integral point set $\mathcal{P}$ is a set of $n$ points in the $m$-dimensional Euclidean space $\mathbb{E}^m$ with pairwise integral distances. The largest occurring distance is called its diameter.
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Figure: Some examples of plane integral point sets.
In radio astronomy systems of antennas are used.

If the distance between two antennas is not an integral multiple of the used wave length interference occurs.

Figure: Very Large Array in New Mexico, USA (Image courtesy of NRAO/AUI)
Definition

We denote the minimum possible diameter of an \( n \)-point integral point set in \( \mathbb{E}^m \) by \( d(m, n) \).

Figure: Plane integral point sets with minimum diameter.
Bounds

Theorem (Harborth, Kemnitz, Möller, 1993)

\[ d(m, n) \leq e^{c \log(n-m) \log \log(n-m)} \]
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Theorem (Solymosi, 2003)

\[ d(2, n) \geq c \cdot n \]
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Theorem (K., Wassermann, 2006)

Plane integral point sets \( \mathcal{P} \) with \( n^\lambda \) collinear points for a fixed \( \lambda, \varepsilon > 0 \) fulfill

\[ \text{diam}(\mathcal{P}) \geq e^{\frac{\lambda}{4 \log 2(1+\varepsilon)} \cdot \log n \cdot \log \log n} \].
Exact values for $d(2, n)$

Theorem (K., Wassermann, 2006)

Exact values for $d(2, n)$ up to $n = 122$. For $9 \leq n \leq 122$ each example consists of $n – 1$ collinear points and one point apart.

Figure: Point set with $n – 1$ collinear points and 1 point apart.
Bounds and exact values for $d(3, n)$

**Theorem (Kanold, 1981)**

$$d(3, n) > \sqrt{\frac{n}{14}} \text{ for } n \geq 5.$$
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**Theorem (Piepmeyer, 1988); (K., 2007)**

$$d(3, 5) = 3, d(3, 6) = 4, d(3, 7) = 8, d(3, 8) = 13,$$
$$d(3, 9) = 16, d(3, 10) = d(3, 11) = d(3, 12) = 17,$$
$$d(3, 13) = 56, d(3, 14) = 65, d(3, 15) = 77,$$
$$d(3, 16) = 86, d(3, 17) = 99, d(3, 18) = 112,$$
$$d(3, 19) = 133, d(3, 20) = 154, d(3, 21) = 195,$$
$$d(3, 22) = 212, d(3, 23) = 228, d(3, 24) = 244.$$
Truncated simplices

Theorem (K., Laue, 2006)
\[ d(m, m^2 + m) \leq 17 \text{ for } m \geq 2. \]

Proof
Consider a regular \( m \)-dimensional simplex with side length 23. Cut a regular simplex with side length 8 on each of the \( m + 1 \) corners.
Theorem (K., Laue, 2006)

\[ d(m, m^2 + m) \leq 17 \quad \text{for} \quad m \geq 2. \]

Proof

Consider a regular \( m \)-dimensional simplex with side length 23. Cut a regular simplex with side length 8 on each of the \( m + 1 \) corners.
Good constructions for integral point sets

A \rightarrow P \rightarrow B

Replace $P_i$ by an $m-1$-dimensional regular simplex with side length 1.

Consider an $m-1$-dimensional sphere $S$ that intersects $P_i$ and has its center on $AB$. Determine an arbitrary integral point set on $S$. If $m=3$ then $S$ is a circle.
Good constructions for integral point sets

(1) Replace $P_i$ by an $m - 1$-dimensional regular simplex with side length 1.
Good constructions for integral point sets

(1) Replace $P_i$ by an $m - 1$-dimensional regular simplex with side length 1.

(2) Consider an $m - 1$-dimensional sphere $S$ that intersects $P_i$ and has its center on $\overline{AB}$. Determine an arbitrary integral point set on $S$. If $m = 3$ then $S$ is a circle.
Brave conjecture

Conjecture
For $n \geq \max(9, m^2 + m + 1)$ integral point sets with minimum diameter can be obtained using the previous two constructions.

Corollary
For $9 \leq n \leq 122$ we have $d(m, n - 2 + m) \leq d(2, n)$. 

Integral point sets in Euclidean spaces
Sascha Kurz
Introduction
Minimum diameter
No-three-on-a-line
No-four-on-a-circle
Semi-general position

**Definition**

An integral point set is in semi-general position if no \(m + 1\) points are contained in a hyperplane. We denote the corresponding minimum diameter by \(\overline{d}(m, n)\).
Semi-general position

Definition

An integral point set is in semi-general position if no $m + 1$ points are contained in a hyperplane. We denote the corresponding minimum diameter by $d(m, n)$.

Theorem (Harborth et. al., 1998); (K., 2006)

For $m = 2$ and $3 \leq n \leq 36$ all points of an example with minimum diameter $d(2, n)$ are situated on a circle.

General position

Definition

An integral point set is in general position if it is in semi-general position and no $m + 2$ points are situated on a hypersphere. We denote the corresponding minimum diameter by $d(m, n)$. 

Theorem (Kemnitz, 1988)

\[ d(2, 3) = 1, \quad d(2, 4) = 8, \quad d(2, 5) = 73, \quad d(2, 6) = 174. \]

Famous question (Erdős)

Are there seven points in the plane, no three on a line, no four on a circle with pairwise integral distances?
General position

Definition
An integral point set is in general position if it is in semi-general position and no \( m + 2 \) points are situated on a hypersphere. We denote the corresponding minimum diameter by \( \hat{d}(m, n) \).

Theorem (Kemnitz, 1988)
\[
\hat{d}(2, 3) = 1, \quad \hat{d}(2, 4) = 8, \quad \hat{d}(2, 5) = 73, \quad \text{and} \quad \hat{d}(2, 6) = 174.
\]
General position

Definition
An integral point set is in general position if it is in semi-general position and no $m + 2$ points are situated on a hypersphere. We denote the corresponding minimum diameter by $\ddot{d}(m, n)$.

Theorem (Kemnitz, 1988)
$\ddot{d}(2, 3) = 1$, $\ddot{d}(2, 4) = 8$, $\ddot{d}(2, 5) = 73$, and $\ddot{d}(2, 6) = 174$.

Famous question (Erdős)
Are there seven points in the plane, no three on a line, no four on a circle with pairwise integral distances?
The late answer

Figure: Integral heptagon in general position \( \rightsquigarrow \)

\[ d(2, 7) = 22270. \]
Thank you very much for your attention.

References

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