Finding extremal voting games via integer linear programming

Sascha Kurz
Business mathematics
University of Bayreuth
sascha.kurz@uni-bayreuth.de

Euro 2012 – Vilnius
Voting games and their properties

"Well - there would have been confusion with the multi-party voting system."
Binary voting games

Definition – Binary voting games
A mapping $\chi : 2^N \rightarrow \{0, 1\}$, where $2^N$ denotes the set of subsets of $N := \{1, 2, \ldots, n\}$. 
Binary voting games

Definition – Binary voting games
A mapping $\chi : 2^N \rightarrow \{0, 1\}$, where $2^N$ denotes the set of subsets of $N := \{1, 2, \ldots, n\}$.

Definition – Simple game
Binary voting game $\chi$ with $\chi(\emptyset) = 0$, $\chi(N) = 1$, and $\chi(S) \leq \chi(T)$ for all $S \subseteq T$. 
### Binary voting games

#### Definition – Binary voting games
A mapping \( \chi : 2^N \to \{0, 1\} \), where \( 2^N \) denotes the set of subsets of \( N := \{1, 2, \ldots, n\} \).

#### Definition – Simple game
Binary voting game \( \chi \) with \( \chi(\emptyset) = 0 \), \( \chi(N) = 1 \), and \( \chi(S) \leq \chi(T) \) for all \( S \subseteq T \).

#### Isbell’s desirability relation
\( i \sqsubseteq j \) for two voters \( i, j \in N \) if and only if \( \chi(\{i\} \cup S\setminus\{j\}) \geq \chi(S) \) for all \( \{j\} \subseteq S \subseteq N\setminus\{i\} \).
Binary voting games

Definition – Complete simple game

Simple game $\chi$ where the binary relation $\sqsupseteq$ is a total preorder, i.e.

1. $i \sqsupseteq i$ for all $i \in N$,
2. $i \sqsupseteq j$ or $j \sqsupseteq i$ (including “$i \sqsupseteq j$ and $j \sqsupseteq i$”) for all $i, j \in N$, and
3. $i \sqsupseteq j, j \sqsupseteq h$ implies $i \sqsupseteq h$ for all $i, j, h \in N$. 
Binary voting games

**Definition – Complete simple game**

Simple game $\chi$ where the binary relation $\sqsubseteq$ is a total preorder, i.e.

1. $i \sqsubseteq i$ for all $i \in N$,
2. $i \sqsubseteq j$ or $j \sqsubseteq i$ (including “$i \sqsubseteq j$ and $j \sqsubseteq i$”) for all $i, j \in N$, and
3. $i \sqsubseteq j, j \sqsubseteq h$ implies $i \sqsubseteq h$ for all $i, j, h \in N$.

**Definition – Weighted voting game**

Simple game (or complete simple game) $\chi$ such that there are weights $w_i \in \mathbb{R}_{\geq 0}$ for all $i \in N$ and a quota $q \in \mathbb{R}_{>0}$ satisfying $\sum_{i \in S} w_i \geq q$ exactly if $\chi(S) = 1$, where $\emptyset \subseteq S \subseteq N$.

Notation: $[q; w_1, w_2, \ldots, w_n]$. 
Example of a weighted voting game

Winning ($\chi(S) = 1$) and losing ($\chi(S) = 0$) coalitions of the weighted voting game $[4; 3, 2, 1, 1]$: 
### Enumeration results

<table>
<thead>
<tr>
<th></th>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$#S$</td>
<td></td>
<td>1</td>
<td>3</td>
<td>8</td>
<td>28</td>
<td>208</td>
<td>16351</td>
<td>$&gt; 4.7 \cdot 10^8$</td>
<td>$&gt; 1.3 \cdot 10^{18}$</td>
<td>$&gt; 2.7 \cdot 10^{36}$</td>
</tr>
<tr>
<td>$#C$</td>
<td></td>
<td>1</td>
<td>3</td>
<td>8</td>
<td>25</td>
<td>117</td>
<td>1171</td>
<td>44313</td>
<td>16175188</td>
<td>284432730174</td>
</tr>
<tr>
<td>$#W$</td>
<td></td>
<td>1</td>
<td>3</td>
<td>8</td>
<td>25</td>
<td>117</td>
<td>1111</td>
<td>29373</td>
<td>2730164</td>
<td>989913344</td>
</tr>
</tbody>
</table>

Tabelle: Number of distinct simple games, complete simple games, and weighted voting games up to symmetry, i.e. orbits under the symmetric group on $n$ elements.
An integer linear programming formulation of a simple game

\[ x_\emptyset = 0 \]  \hspace{1cm} (1)

\[ x_N = 1 \]  \hspace{1cm} (2)

\[ x_S \leq x_T \hspace{1cm} \forall S \subseteq T \subseteq N \]  \hspace{1cm} (3)

\[ x_S \in \{0, 1\} \hspace{1cm} \forall S \subseteq N \]  \hspace{1cm} (4)
How to measure power

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td>A</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>C</td>
<td>D</td>
<td>B</td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>B</td>
<td>D</td>
<td>C</td>
</tr>
<tr>
<td>D</td>
<td>A</td>
<td>C</td>
<td>B</td>
<td>D</td>
</tr>
</tbody>
</table>

**Shapley-Shubik power index** (there are more power indices)

The power of a player is measured by the fraction of the possible voting sequences in which that player casts the deciding vote, that is, the vote that first guarantees passage or failure. The power index is normalized between 0 and 1.

Example: $A \rightarrow 3, B \rightarrow 2, C \rightarrow 1, D \rightarrow 1, \text{quota} = 4$

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>ABCD</td>
<td>ABDC</td>
<td>ACBD</td>
<td>ACDB</td>
<td>ADBC</td>
</tr>
<tr>
<td>BACD</td>
<td>BADC</td>
<td>BCAD</td>
<td>BCDA</td>
<td>BDAC</td>
</tr>
<tr>
<td>CABD</td>
<td>CADB</td>
<td>CBAD</td>
<td>CBDA</td>
<td>CDAB</td>
</tr>
<tr>
<td>DABC</td>
<td>DACB</td>
<td>DBAC</td>
<td>DBCA</td>
<td>DCAB</td>
</tr>
</tbody>
</table>

$Pow(A) = \frac{12}{24}, Pow(B) = \frac{4}{24}, Pow(C) = \frac{4}{24}, Pow(D) = \frac{4}{24}$
Problem 1

Find a voting game whose Shapley-Shubik vector has minimal $L_1$-distance to a given ideal power distribution $\sigma$, e.g.

$$\sigma_n = \left( \frac{2}{2n-1}, \frac{2}{2n-1}, \cdots, \frac{2}{2n-1}, \frac{1}{2n-1} \right).$$
Inverse Power Index Problem

Problem 1

Find a voting game whose Shapley-Shubik vector has minimal $L_1$-distance to a given ideal power distribution $\sigma$, e.g.

$$\sigma_n = \left(\frac{2}{2n-1}, \frac{2}{2n-1}, \ldots, \frac{2}{2n-1}, \frac{1}{2n-1}\right).$$

ILP approach

- $w_S = (|S|! \cdot (n - |S| - 1)!)/n!$ (constants for the Shapley-Shubik index)
- $x_S \in \{0, 1\}$: is coalition $S \subseteq N$ winning?
- $y_{i,S} \in \{0, 1\}$: is coalition $S$ a swing for voter $i$?
- $p_i$: Shapley-Shubik power for voter $i$
- $d_i$: $|p_i - \sigma_i|$
An ILP formulation for the inverse Shapley-Shubik problem

\[
\begin{align*}
\min & \quad \sum_{i=1}^{n} d_i \\
\text{s.t.} & \quad \sigma_i - d_i \leq p_i \leq \sigma_i + d_i & \forall i \in N, \\
& \quad p_i = \sum_{S \in N \setminus \{i\}} w_S \cdot y_{i,S} & \forall i \in N, \\
& \quad y_{i,S} = x_{S \cup \{i\}} - x_S & \forall i \in N, \quad S \subseteq N \setminus \{i\}, \\
& \quad x_S \geq x_{S \setminus \{j\}} & \forall \emptyset \neq S \subseteq N, \quad j \in S, \\
& \quad x_\emptyset = 0 & \forall S \subseteq N, \\
& \quad x_N = 1 & \\
& \quad x_S \in \{0, 1\} & \forall S \subseteq N, \\
& \quad y_{i,S} \in \{0, 1\} & \forall i \in N, \quad S \subseteq N \setminus \{i\}, \\
& \quad d_i, p_i \geq 0 & \forall i \in N.
\end{align*}
\]
Results – Minimal deviations

<table>
<thead>
<tr>
<th>$n$</th>
<th>$| \cdot |_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.000000</td>
</tr>
<tr>
<td>2</td>
<td>0.333333</td>
</tr>
<tr>
<td>3</td>
<td>0.266667</td>
</tr>
<tr>
<td>4</td>
<td>0.214286</td>
</tr>
<tr>
<td>5</td>
<td>0.144444</td>
</tr>
<tr>
<td>6</td>
<td>0.096970</td>
</tr>
<tr>
<td>7</td>
<td>0.084249</td>
</tr>
<tr>
<td>8</td>
<td>0.761905</td>
</tr>
<tr>
<td>9</td>
<td>0.658263</td>
</tr>
<tr>
<td>10</td>
<td>0.054386</td>
</tr>
<tr>
<td>11</td>
<td>0.050216</td>
</tr>
<tr>
<td>12</td>
<td>0.046377</td>
</tr>
<tr>
<td>13</td>
<td>0.042937</td>
</tr>
<tr>
<td>14</td>
<td>0.037759</td>
</tr>
</tbody>
</table>

Tabelle: Optimal deviation for $\sigma_n$ in the set of complete simple games.
A simple game $\chi$ is $\alpha$-roughly weighted if there are weights $w_i \in \mathbb{R}_{\geq 0}$ such that

1. $\sum_{i \in S} w_i \geq 1$ if $\chi(S) = 1$;
2. $\sum_{i \in S} w_i \leq \alpha$ if $\chi(S) = 0$

for all $\emptyset \subseteq S \subseteq N$.

The critical threshold value $\mu(\chi)$ of $\chi$ is the minimum value $\alpha \geq 1$ such that $\chi$ is $\alpha$-roughly weighted.
**α-roughly weighted games**

### Definition (Gvozdeva, L. Hemaspaandra, Slinko; 2012)

A simple game $\chi$ is $\alpha$-roughly weighted if there are weights $w_i \in \mathbb{R}_{\geq 0}$ such that

1. $\sum_{i \in S} w_i \geq 1$ if $\chi(S) = 1$;
2. $\sum_{i \in S} w_i \leq \alpha$ if $\chi(S) = 0$

for all $\emptyset \subseteq S \subseteq N$.

The critical threshold value $\mu(\chi)$ of $\chi$ is the minimum value $\alpha \geq 1$ such that $\chi$ is $\alpha$-roughly weighted.

### Problem 2

What is the maximal critical threshold value of a complete simple game on $n$ voters?
LP formulation for the critical threshold value

\[ \mu(\chi) = \min \alpha \]

\[ w(S) \geq 1 \quad \forall S \subseteq N : \chi(S) = 1 \]

\[ w(S) \leq \alpha \quad \forall S \subseteq N : \chi(S) = 0 \]

\[ \alpha \geq 1 \]

\[ w_1, \ldots, w_n \in \mathbb{R}_{\geq 0} \]

Remark

The conditions can be restricted to minimal winning and maximal losing coalitions.
Using duality

General linear program

\[
\begin{align*}
\text{max} & \quad c^T x \\
Ax & \leq b \\
x & \geq 0
\end{align*}
\]

... as a feasibility problem

\[
\begin{align*}
c^T x &= b^T y \\
Ax & \leq b \\
A^T y & \geq c \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]
An ILP approach for the determination of the maximum critical threshold value

\[
\begin{align*}
\text{max} & \quad \sum_{S \subseteq N} u_S \\
x_{\emptyset} &= 1 - x_N = 0 \\
x_S - x_{S\setminus\{i\}} &\geq 0 \quad \forall \emptyset \neq S \subseteq N, i \in S \\
\sum_{\{i\} \subseteq S \subseteq N} u_S - \sum_{\{i\} \subseteq T \subseteq N} v_T &\leq 0 \quad \forall i \in N \\
\sum_{T \subseteq N} v_T &\leq 1 \\
u_S - x_S &\leq 0 \quad \forall S \subseteq N \\
v_T + x_T &\leq 1 \quad \forall T \subseteq N \\
x_S &\in \{0, 1\} \quad \forall S \subseteq N \\
u_S, v_S &\geq 0 \quad \forall S \subseteq N
\end{align*}
\]
The maximum critical threshold value $s(n)$ of a complete simple game

<table>
<thead>
<tr>
<th>n</th>
<th>s(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-6</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{26}{21}$</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{4}{3}$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{38}{27}$</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{22}{15}$</td>
</tr>
<tr>
<td>12</td>
<td>$\frac{14}{9}$</td>
</tr>
<tr>
<td>13</td>
<td>$\frac{33}{20}$</td>
</tr>
<tr>
<td>14</td>
<td>$\frac{111}{64}$</td>
</tr>
<tr>
<td>15</td>
<td>$\frac{123}{68}$</td>
</tr>
<tr>
<td>16</td>
<td>$\frac{15}{8}$</td>
</tr>
</tbody>
</table>
Local monotonicity of the Public Good Index

Definition – PGI

Given a simple game $\chi$. A coalition $S$ called minimal winning coalition if $\chi(S) = 1$ but all proper subsets of $S$ are losing. By $MW_i$ we denote the number of minimal winning coalitions containing player $i$. The Public Good Index (PGI) for player $i$ is given by

$$PGI(i) = \frac{MW_i}{\sum_{j \in N} MW_j}.$$
Local monotonicity of the Public Good Index

Definition – PGI

Given a simple game $\chi$. A coalition $S$ called minimal winning coalition if $\chi(S) = 1$ but all proper subsets of $S$ are losing. By $MW_i$ we denote the number of minimal winning coalitions containing player $i$. The Public Good Index (PGI) for player $i$ is given by

$$PGI(i) = \frac{MW_i}{\sum_{j \in N} MW_j}.$$ 

Problem 3

Let $i \sqsupset j$ in a complete simple game $\chi$.

How large can $MW_i - MW_j$ be?
An ILP formulation

**Variables**

\( y_S \in \{0, 1\} \) with \( y_S = 1 \) is coalition \( \emptyset \subseteq S \subseteq N \) is a minimal winning coalition.

**Inequalities**

\[
\begin{align*}
y_S &\leq x_S \quad \forall S \subseteq N \\
y_S &\geq 1 - X_{S\setminus\{\min i : i \in S\}} \quad \forall \emptyset \neq S \subseteq N \\
y_S &\leq 1 - X_{S\setminus\{i\}} \quad \forall \emptyset \neq S \subseteq N, \ i \in S \\
y_S &\in \{0, 1\} \quad \forall \emptyset \neq S \subseteq N \\
MW_i &= \sum_{\{i\} \subseteq S \subseteq N} y_S \quad \forall i \in N
\end{align*}
\]
Thank you very much for your attention!

Conclusion
Whenever a certain class of voting games should be analyzed concerning a certain parameter, the extremal values may be obtained using integer linear programming techniques – enlarging the scope of exhaustive search methods.

Proposals are highly welcome.