Regular matchstick graphs

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Matchstick graphs
Definition (Matchstick graph)

Graph $G = (V, E)$ such that

- there exists an injective embedding $f : V \rightarrow \mathbb{R}^2$ with $\|f(v_1) - f(v_2)\|_2 = 1$ for all $\{v_1, v_2\} \in E$ and
- the corresponding edges are non-crossing.
Regular matchstick graphs

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- the corresponding edges are non-crossing.

Definition
A graph is $r$-regular if every vertex has degree $r$. 
Observation
Every finite planar graph contains a vertex of degree at most 5. Thus for $r \geq 6$ no finite $r$ regular matchstick graph exists.
Regular matchstick graphs

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Remark
There exist $r$-regular matchstick graphs for $r \in \{0, 1, 2, 3, 4\}$ consisting of 1, 2, 3, 8 or 52 vertices.
Does a (finite) 5-regular matchstick graph exist?

Prologue

- Open problem at an Oberwolfach meeting
- Preprint claiming the non-existence proof turned out to be incorrect
- Some places in the www or in older literature refer to this non-existence proof
Does a (finite) 5-regular matchstick graph exist?

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In this talk

...we prove the non-existence of a finite 5-regular matchstick graph.
Preliminaries

Definition
By $F_i$ we denote the number of $i$-gons of a given planar graph.

Lemma
For a finite 5-regular planar graph we have

$$\sum_{i=3}^{\infty} (10 - 3i)F_i = F_3 - 2F_4 - 5F_5 - 8F_6 - 11F_7 - \cdots = 20.$$
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Definition (face set)
Multiset $f(v)$ containing the number of corners of the adjacent faces of a vertex $v$. $\rightsquigarrow f(v) = \{3, 3, 3, 4, 4\}$ in the example (blue vertex).
Preliminaries

Definition (contribution)

• $c(a) := \frac{10-3a}{a}$ for $a \in \mathbb{N}$
• $c(v) := \sum_{i=1}^{5} c(a_{v,i}) = \sum_{i=1}^{5} \frac{10-3a_{v,i}}{a_{v,i}}$ for a vertex $v$ with $f(v) = \{a_{v,1}, \ldots, a_{v,5}\}$

Lemma

For a finite 5-regular planar graph $G = (V, E)$ we have $\sum_{v \in V} c(V) = 20$. 
Preliminaries

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Lemma

For a finite 5-regular planar graph \( G = (V, E) \) we have

\[ \sum_{v \in V} c(V) = 20. \]
Lemma
The face sets with non-negative contribution are given by

\[ c(\{3,3,3,3,4\}) = \frac{5}{6}, \quad c(\{3,3,3,4,4\}) = 0, \]
\[ c(\{3,3,3,3,5\}) = \frac{1}{3}, \quad c(\{3,3,3,3,6\}) = 0. \]
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Proof strategy
Partition \( V \) into subsets, each having non-negative contribution
\[ c(U) = \sum_{u \in U} c(u). \]
**$\mathcal{T} Q$-classes**

**Definition**

Equivalence relation $\sim$ on the set $\mathcal{T} Q$ of triangles and quadrangles with angles in $\left\{ \frac{1}{3}\pi, \frac{2}{3}\pi \right\}$:

- if $x, y \in \mathcal{T} Q$ share an edge we require $x \sim y$
- take the transitive closure
- $\mathcal{T} Q$-class $x \sim = \{ y \in \mathcal{T} Q \mid y \sim x \}$
**$\mathcal{TQ}$-classes**

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**Example**
Lemma

$\mathcal{TQ}$-class $\mathcal{B}$:

1. faces are edge-to-edge connected
2. vertices are situated on a suitable regular triangular lattice
$\mathcal{TQ}$-classes

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2. vertices are situated on a suitable regular triangular lattice

Parameters of a $\mathcal{TQ}$-class $\mathcal{B}$

- $k$: number of outer edges (exactly one adjacent face is in $\mathcal{B}$)
- $\tau$: number of missing valences
- $\sigma$: contribution of the triangles and quadrangles in $\mathcal{B}$
- $b_2$: number of inner angles $\frac{2\pi}{3}$ at an outer vertex (not all adjacent faces are in $\mathcal{B}$)
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- \( \sigma \): contribution of the triangles and quadrangles in \( B \)
- \( b_2 \): number of inner angles \( \frac{2\pi}{3} \) at an outer vertex (not all adjacent faces are in \( B \))
- \( b_1 \): number of vertices of special type:
Parameters of a $TQ$-class $B$

Example

<table>
<thead>
<tr>
<th></th>
<th>green</th>
<th>red</th>
<th>blue</th>
<th>brown</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>10</td>
<td>5</td>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>$\tau$</td>
<td>20</td>
<td>11</td>
<td>9</td>
<td>22</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>16</td>
</tr>
<tr>
<td>$b_1$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$b_2$</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>
**Definition**

A $\mathcal{TR}$-class is a union of triangles and quadrangles on a triangular grid.
TR-classes

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Theorem
For a TR-class we have
\[ \sigma - k + \frac{\tau - k}{3} + \frac{5}{3}b_1 + \frac{5}{3}b_2 = 0. \]
An identity for $\mathcal{TR}$-classes

Theorem
For a $\mathcal{TR}$-class we have

$$\sigma - k + \frac{\tau - k}{3} + \frac{5}{3} b_1 + \frac{5}{3} b_2 = 0.$$ 

Proof
By induction: adding a triangle or a quadrangle (new, old).
Contribution of a $TQ$-class $B$

Definition
If $f(v) = \{a_v,1, \ldots, a_v,5\}$ is the face set of vertex $v$ there are five corresponding face arcs $[v,1], \ldots, [v,5]$. 

Strategy
To every face angle $[v,i]$, where $v \in B$, we assign a weight $\omega([v,i]) \geq c([v,i])$ which fulfills $\sum_{v \in B} \sum_{i=1}^{5} \omega([v,i]) \leq 0$.

- If the face $([v,i]$ is in $B$, then $\omega([v,i]) = c([v,i])$. 


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- if the face $(v, i)$ is in $\mathcal{B}$, then $\omega([v, i]) = c([v, i])$
Assignment of the weights $\omega$

- if the face $(v, i)$ is in $B$, then $\omega([v, i]) := c([v, i])$
- if the none of the two edges corresponding to face arc $[v, i]$ is an outer edge, then $\omega([v, i]) := \frac{1}{3}$
Assignment of the weights $\omega$

- if the face $(v, i)$ is in $\mathcal{B}$, then $\omega([v, i]) := c([v, i])$
- if the none of the two edges corresponding to face arc $[v, i]$ is an outer edge, then $\omega([v, i]) := \frac{1}{3}$
- for the remaining face angles we consider the outer edges (red, blue)
Assignment of the weights $\omega$ - outer edges

If the outer edges form a cycle $C$

- $\omega([v, i]) := -1$ if $C$ is a pentagon
- $\omega([v, i]) := -\frac{4}{3}$ if $C$ has at least six edges.
Assignment of the weights $\omega$ - outer edges

If the outer edges form a cycle $C$ then we set

- $\omega([v, i]) := -1$ if $C$ is a pentagon and
- $\omega([v, i]) := -\frac{4}{3}$ if $C$ has at least six edges.
Possible problems

Vertices contained in more than one $\mathcal{T}Q$-class:

\[ \sum_{i=1}^{\nu} \omega_1(v, i) + \omega_2(v, i) \geq \sum_{i=1}^{\nu} c(v, i). \]
Possible problems

Vertices contained in more than one $\mathcal{T}Q$-class:

\[
\sum_{i=1}^{5} \omega_1([v, i]) + \omega_2([v, i]) \geq \sum_{i=1}^{5} c([v, i]).
\]
The proof

Lemma
For $TQ$-classes $B_i$ of a 5-regular matchstick graph we have

$$c\left(\bigcup_i B\right) \leq 0.$$
The proof

Lemma
For $TQ$-classes $B_i$ of a 5-regular matchstick graph we have

$$c\left(\bigcup_i B\right) \leq 0.$$

Theorem
No finite 5-regular matchstick graph exists.

Proof
Let $C$ be the union of all $TQ$-classes of $G = (V, E)$, then

$$c(V) = c(C) + \sum_{v \in V \setminus C} c(v) \leq 0 + \sum_{v \in V \setminus C} 0 = 0.$$
The end

Infinite 5-regular match stick graph:
The end

Infinite 5-regular match stick graph:

Thank you very much for your attention!