Integral point sets over $\mathbb{Z}^m_n$

Sascha Kurz
sascha.kurz@uni-bayreuth.de
www.wm.uni-bayreuth.de

University of Bayreuth

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Integral point sets

**Definition**

An integral point set $\mathcal{P}$ is a set of $n$ points in the $m$-dimensional space $\mathbb{K}^m$ with pairwise integral distances. The largest occurring distance is called its diameter.
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Figure: Some examples for $\mathbb{K} = \mathbb{R}$, $m = 2$ in the Euclidean metric.
Applications in radio astronomy

In radio astronomy systems of antennas are used.

If the distance between two antennas is not an integral multiple of the used wave length interference occurs.

Normalization $\leadsto$ integral point sets

Figure: Very Large Array in New Mexico, USA (Image courtesy of NRAO/AUI)
Integral point sets over $\mathbb{Z}_n^m$

Introduction

Integral point sets in $\mathbb{E}^m$

**Definition**

We denote the minimum diameter of an $n$-point integral point set in the $m$-dimensional Euclidean space $\mathbb{E}^m$ by $d(n, m)$.

Theorem (Harborth, Kemnitz, Möller, 1993)

\[ d(n, m) \leq e^{c \log (n - m) \log \log (n - m)} \]

Theorem (Solymosi, 2003)

\[ d(n, 2) \geq c \cdot n \]

Theorem (K., Wassermann, 2006)

Exact values for $d(n, 2)$ up to $n = 122$. For $9 \leq n \leq 122$ each example consists of $n - 1$ collinear points and one point apart.
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**Figure**: Point set with $n - 1$ collinear points and 1 point apart.
Integral point sets in $\mathbb{F}^2$

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Theorem (K., Wassermann, 2006)

Plane integral point sets $P$ with $n^\lambda$ collinear points for a fixed $\lambda$, $\varepsilon > 0$ fulfill

$$\text{diam}(P) \geq n^{\lambda \log 2/(1+\varepsilon)} \log \log n.$$
Further conditions

Definition

An $m$-dimensional point set $\mathcal{P}$ is in semi-general position if no $m + 1$ points are contained in a hyperplane.
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Problem (Erdős)

Are there seven points in the plane with pairwise integral distances no three in a line, no four on a circle?

Partial answer

Incorrect proofs of non-existence. Diameter of such a set must be greater 20 000 (K., Wassermann 2006).
Integral point sets in $\mathbb{Z}^m$ (Cluster)

**Definition**

An $m$-dimensional integral point set $\mathcal{P}$ in general position consisting of $n$ points where also the coordinates of the points are integral is called an $n_m$-cluster.
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Do $7_2$-cluster exist? More generally, are there any $n_m$-cluster for $n \geq m + 5$?

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Main idea

There are some difficult open problems for integral point sets in $\mathbb{E}^m$ and $\mathbb{Z}^m$. To gain some insight we apply the homomorphism

$$\phi_n : \mathbb{Z} \rightarrow \mathbb{Z}_n, \quad x \mapsto x + \mathbb{Z}n =: \bar{x}.$$
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Two points $(u_1, \ldots, u_m), (v_1, \ldots, v_m) \in \mathbb{Z}_n^m$ are at integral distance if $\exists \, d \in \mathbb{Z}_n$ with $\sum_{i=1}^{m} (u_i - v_i)^2 = d^2$. 

An integral point set in $\mathbb{Z}_n^m$ is a set of $n$ points where all pairs of points are at integral distance. The maximum number of points in $\mathbb{Z}_n^m$ with pairwise integral distances is denoted by $I(n, m)$. 
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- An integral point set in $\mathbb{Z}_n^m$ is a set of $n$ points where all pairs of points are at integral distance.
- The maximum number of points in $\mathbb{Z}_n^m$ with pairwise integral distances is denoted by $\mathcal{I}(n, m)$.
Integral point sets over $\mathbb{Z}_n^m$

Main part

Values of $\mathcal{I}(n, m)$

Observation

$\mathcal{I}(n, 1) = n$, $\mathcal{I}(1, m) = 1$, and $\mathcal{I}(2, m) = 2^m$. 
Integral point sets over $\mathbb{Z}_n^m$

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Table: Values of $\mathcal{I}(n, m)$ for small parameters $n$ and $m$. 
Integral point sets over $\mathbb{Z}_n^m$ 

Main part

Homomorphisms and Hamming distances

Observation
The integral point sets over $\mathbb{Z}_3^m$ correspond to subsets of $\mathbb{Z}_3^m$ with Hamming distances $h(u, v) \not\equiv 2 \mod 3$. 

Theorem
For an odd integer $n$ we have $I(2n, m) = 2^m \cdot I(n, m)$. 

Lemma
For two coprime integers $a$ and $b$ we have $I(a \cdot b, m) \geq I(a, m) \cdot I(b, m)$. 

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Integral point sets over $\mathbb{Z}_n^m$

Main part

Integral point sets over the plane $\mathbb{Z}_n^2$

Lemma

- If the prime factorization of $n$ is given by $n = \prod_{i=1}^{s} p_i^{r_i}$ we have $\mathcal{I}(n, 2) \geq n \cdot \prod_{i=1}^{s} p_i^{\lfloor r_i/2 \rfloor}$.

- If the prime factorization of $n$ is given by $n = 2 \cdot \prod_{i=2}^{s} p_i^{r_i}$ we have $\mathcal{I}(n, 2) \geq 2n \cdot \prod_{i=2}^{s} p_i^{\lfloor r_i/2 \rfloor}$.

- The corresponding sets are given by $\{(u, vk) \mid u, v \in \mathbb{Z}_n\}$ for some suitable $k \in \mathbb{Z}_n$. 

Conjecture

The above bounds are tight. (Verified for $n \leq 100$.)
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In the case of finite geometries point sets in semi-general position are called arcs in the case of planes or caps in the more general higher dimensional case.
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Corollary
$$\mathcal{I}(p, 2) \leq p + 1$$ for odd primes $p$.

Conjecture
$$\mathcal{I}(p, 2) = \frac{p-1}{2}$$ for $p \equiv 1 \mod 4$ and
$$\mathcal{I}(p, 2) = \frac{p+1}{2}$$ for $p \equiv 3 \mod 4$. 
General position in the plane

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